SATVAT JAGWANI
Cracking the JEE ADVANCED

2015 SOLVED PAPER
AMU ENGINEERING

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editorial

Mathematics - A Perennial Source of Joy

Mathematical puzzles always fascinate children. Some answers depend on intelligence. Some needed fast calculations. As all students are like children for the seniors, let us see some interesting questions. A very simple question is "What is $i + i$ "? Then start if a current 2 amp is flowing through resistances connected in series? What will be answer if the current of 2 amp is flow through r_1, and r_2? Will it be the same value in parallel connection and so on. What is 1 + 1 + 1 in decimal system? In binary system?

Never take an answer for granted. Learn the matrix method and where do you apply it. The applications of coordinate geometry are very interesting and more important in the present day applications than the Euclid's geometry. Differential equations are important in the study of many subjects. Astronomy is very important if you want to be a pilot or space scientist. For Marine engineers, it is equally important. Now one can get anything from internet. But remember that a computer scientist is behind it. One has the most powerful computers in the national laboratories. But the scientist behind the machine decides what to calculate and to what degree of accuracy.

In automatic analysers such as X-ray spectrometers, there are automatic analysers once a sample is loaded. The software depends on the supplier. A good research worker should disconnect the software that is supplied and develop one's own method of analysis. Students should try to develop the most powerful software-Human Intelligence.

Anil Ahlawat
Editor

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ALERT
AWARENESS CLIPS

SIMfully Yours: Cheating in exams goes hi-Tech

A tabiz is worn for good luck. Some bring more luck than the others. These are the ones that come with earpieces and SIM cards to help students cheat in exams. Stores stocked with such gadgets are operating openly in the Capital and cater to anyone willing to shell out Rs. 8,000 - 15,000 for the equipment.

The devices come with a warranty and can pass through X-ray scanners undetected, or so claim the manufacturers. The Tabiz comes with an in-built SIM card. In a demo of sorts, the helper made one of the reporters, posing as a student wear the tabiz and, much to his surprise, tossed a dot-sized audio device into his ear. The other reporter was asked to step out and call a number. The SIM card in the tabiz picked up the call which was answered automatically in two seconds. The earpiece was so small that it had to be pulled out of the ear using a magnet.

The device gets automatically activated when a call is made to it. Even a person sitting in another state can make calls and communicate with you. A new SIM card is required for this. The audio quality too was clear.

A store in south Delhi's Green park boasted of having "made many doctors in Bihar". "There would be no problem and no one would get to know about it. We have made many doctors in Bihar using these products so don't worry about the effi ciency. We only sell genuine products with a one-year warranty. The products are so sophisticated that they can't be detected even during the search before boarding a Metro train," said the attendant of the store which operates out of a residential society in the posh area.

Many offer to customize the gadget to fit it into salwar - kameez sets, sports bras, jackets, shirts and regular buttons. They guarantee same-day delivery too.

"The microphone can be fi tted anywhere from your watch to a hair band. While the religious lockets attract less attention, an innerwear can be a perfect option too, hence they remain the top-selling products," the shop manager in Patel Nagar said.

The business is thriving in the digital world too. A simple Google search will throw up a plethora of websites providing such gadgets with an assurance of helping you pass exams without studying. "Pass exams successfully without prior preparation," says the websites of a store in Dwarka, claiming to be in the business for at least 10 years.

According to the police, applicants accused in the medical entrance test bought the cheating devices from a shop in New Delhi.

Courtesy : The Hindustan Times

Bombay top choice of IIT kids, Delhi next

65 Of Top 100 Rankers Opt For Powai

Sixty-five of the top 100 rankers in JEE (Advanced) this year have picked IIT-Bombay, helping it retain its position as the most sought-after campus ahead of 17 other premier Indian Institutes of Technology. The fi gure from the fi rst allotment round marked an increase over last year, which saw 58 among the top 100 opting for the Powai institute. In fact, 2014 was the fi rst time in years that fewer than 60 of the top 100 had picked Powai.

IIT-Delhi emerged as the second choice for aspirants, with 30 students choosing the campus, a drop from last year's 36. IIT-Madras and IIT-Kanpur were a distant third and fourth, with three and two students, respectively.

Of the top 100 JEE (Advanced) rankers, 34 belonged to the IIT-Bombay zone and, therefore, the institute was an obvious choice. Interestingly, many of the 28 rankers from the IIT-Madras zone seem to have opted for IIT-B as only three chose the former.

"A majority of students perceive it as the top institute. At 18 most students make their choices based on what their friends and family would recommend. Computer science and engineering (CSE) and electrical engineering were the popular choices for students in the top ranks," said Devang Khakhar, IIT-B director.

The opening and closing ranks for CSE at IIT-B were 1 and 59, indicating that most in the top 60 have chosen the programme. At IIT-Delhi, it opened at 31 and closed at 102. At Kanpur it started at 26, and 61 in Chennai. Admission to the electrical engineering programme in IIT-B opened at rank 9 and closed at 240. Electrical and mechanical engineering followed CSE as the sought-after courses at Powai and Delhi.

If the old favourites—Madras and Kanpur—have slid in the rankings, Kharagpur and the newer ones, Roorkee and Guwahati, have not managed to get even a single student from the top ranks. IIT offi cials attributed this to the low representation of students from the North-East.

"Students' choices indicate that they do not pick institutes just for academics and campus life, but a lot of them are infl uenced by what is in store outside the campus too, " said former IIT-Delhi director R K Shevgaonkar.

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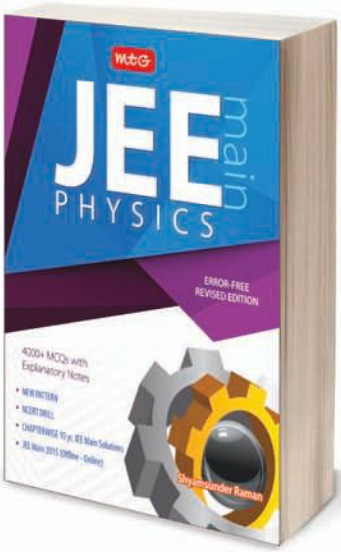

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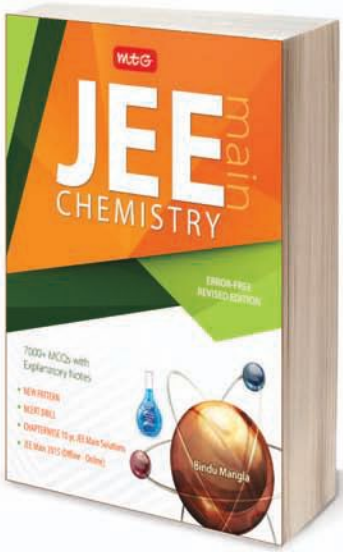

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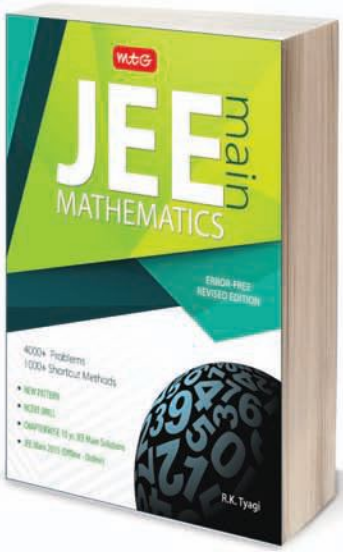

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Cracking the JEE Advanced EXAM

Satvat Jagwani

JEE Advanced 2015 Topper Interview : How Satvat Jagwani scored 469 out of 504?

The topper from Satna, Madhya Pradesh believes enjoying his studies gave him an edge during preparation as well as when the final result for JEE Advanced was declared. "Physics and Mathematics are two of my favorite subjects. So, I did not really feel like I was preparing," he says. In fact, Satvat's future plan includes doing research based work in Physics.

Satvat shares his simple yet effective preparation strategy and how an active support system of parents and teachers helped him crack the toughest engineering entrance exam in India.

Congratulations for your outstanding performance in JEE Advanced 2015! How did you celebrate your success?

Satvat: Thank you so much! It feels awesome to be the topper. I am very grateful to my parents as well as my teachers. It would not have been possible without them.

Were you sure of cracking the prestigious JEE Advanced 2015 exam? How did you make it possible?

Satvat: Honestly, I was not just confident of cracking JEE Advanced, I also knew there was a possibility of me emerging as a topper. You need to put in a lot of effort in an exam of such significance. I am glad my hard work paid off.

How did you find the difficulty level of JEE Advanced 2015 exam?

Satvat: It was not that tough for me. I had prepared well so I was able to solve most of the questions without much difficulty.

What was your score in JEE Main? Which one do you think was tougher Main or Advanced?

Satvat: In JEE Main my score was 330 out of 360. Actually, that inspired me to work harder for JEE Advanced. Between Main and Advanced- none was tough for me. But if I had to choose, I would say that Main had many complicated questions compared to Advanced. Advanced was much simpler for me. Also, JEE Main was lengthier than the Advanced paper.

How did you make it possible? What was your preparation strategy and routine study period for JEE Advanced?

Satvat: I did not really follow any strict study routine. I kept it simple. I worked on the areas I was weak in and focused on strengthening my strong areas. Physics and Mathematics are two of my favorite subjects. So, I did not really feel like I was preparing. I was just enjoying studying. Besides, my teachers helped me a lot. All I had

Satvat Jagwani shared few pointers with students aspiring to achieve a spot on the coveted toppers' list. Keeping it simple, it seems, is what helped him clinch the top spot—attending classes, completing all the homework, clearing doubts in advance and revising the notes are the keys to success he listed.

Regarding his schedule for preparing for the exam, he said although he did not follow any fixed schedule, he spent around five to six hours studying daily after his routine classes. Revealing the secret behind his speed in solving mathematical problems, Jagwani said he learnt Mathematical Abacus, which helped him reduce the amount of rough work required in the examination

to do was revise whatever they would teach me and do my homework for the class.

Yes, I even took lot of mock tests. That helps you to manage the time. They also help you to know where you are making mistakes. I had done well in most of my mock tests. So, that fuelled my confidence for the actual examination.

Which subject was the easiest and which was the toughest according to you in the JEE Advanced exam and during the preparation period?

Satvat: I did not find any subject too easy or too tough. But as I mentioned earlier, I enjoy Physics and Mathematics. So, it was easier for me to prepare for the two areas.

Which IIT do you want to join? Also, have you decided upon your area of specialization?

Satvat: Well, I want to join IIT Bombay. I am thinking about taking up Computer science as my specialization.

That is an interesting choice. So, what are your plans for the future?

Satvat: Thank you. I have not thought a lot about that. But yes, I am interested in doing research work. Possibly, in Physics and Mathematics.

That was all about studies. Tell us what you do for recreation?

Satvat: Honestly, I haven't had much time for recreation in the last few months. But yes, I do enjoy solving Sodoku. It is not only relaxing but also helps stimulating your brain.

Please share your valuable tips and message for JEE aspirants, who will be appearing for the entrance exam next year.

Satvat: Focus on your strengths. Keep practicing and keep revising. To achieve anything you need to work hard.

Wish you all the best for your future!

Satvat: Thanks a lot!

Courtesy : careers360.com

- The cut-off score for JEE (Advanced) has been lowered this year. The candidates who pass the JEE (Main) examination are eligible to appear for the JEE (Advanced) exam, for admissions to Indian Institutes of Technology (IITs). Last year the cut-off was 115 out of a total of 360, this year it is ten marks lower at 105. The cut-off is calculated on the basis of the performance of the candidates who are eligible to appear for JEE (Advanced).

- A total of 12,92,711 candidates had taken the JEE (Main) exam, out of which 1,52,401 are eligible to appear for the JEE (Advanced) exam; based on the cut-off of 105 marks. The exam was conducted in both online and offline modes. Approximately 11 lakh candidates appeared for the offline exam and the remaining 1.8 lakh candidates took the exam through online mode.

IIT-JEE Advanced 2015
Zone wise Toppers List (Top 7)

IIT Bombay	Bharat Khandelwal (AIR 5)
IIT Delhi	Janak Agrawal (AIR 2)
IIT Guwahati	Ayush Ranjan (AIR 94)
IIT Kanpur	Satvat Jagwani (AIR 1)
IIT Kharagpur	Rohit Kumar Jena (AIR 57)
IIT Madras	Kamana Nagendra Reddy (AIR 4)
IIT Roorkee	Himanshu Gupta (AIR 7)

MATHS MUSING

Maths Musing was started in January 2003 issue of Mathematics Today with the suggestion of Shri Mahabir Singh. The aim of Maths Musing is to augment the chances of bright students seeking admission into IITs with additional study material.
During the last 10 years there have been several changes in JEE pattern. To suit these changes Maths Musing also adopted the new pattern by changing the style of problems. Some of the Maths Musing problems have been adapted in JEE benefitting thousand of our readers. It is heartening that we receive solutions of Maths Musing problems from all over India.
Maths Musing has been receiving tremendous response from candidates preparing for JEE and teachers coaching them. We do hope that students will continue to use Maths Musing to boost up their ranks in JEE Main and Advanced.

Prof. Dr. Ramanaiah Gundala, Former Dean of Science and Humanities, Anna University, Chennai

PROBLEM Set 152

JEE MAIN

1. Of the students attending a school party, 60% of the students are girls, 40% of the students like to sing. After these students are joined by 20 more boy students, all of whom like to sing, the party is now 58% girls. How many students now at the party like to sing?

(a) 248 (b) 250 (c) 252 (d) 256

2. P is a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with foci S and S_1 and eccentricity e. The locus of the incentre of triangle PSS_1 is a conic of eccentricity

(a) $\frac{1-e}{1+e}$ (b) $\sqrt{\frac{1-e}{1+e}}$

(c) $\sqrt{\frac{e}{1+e}}$ (d) $\sqrt{\frac{2e}{1+e}}$

3. In triangle ABC, if a, b, c are in A.P. and $C-A=\frac{\pi}{2}$, then $\cos B =$

(a) $\frac{1}{2}$ (b) $\frac{2}{3}$ (c) $\frac{3}{4}$ (d) $\frac{4}{5}$

4. The area bounded by the curve $x^{2/3}+y^{2/3}=1$ is

(a) $\frac{3\pi}{2}$ (b) $\frac{3\pi}{4}$ (c) $\frac{3\pi}{8}$ (d) $\frac{5\pi}{16}$

5. If $C_r=\binom{30}{r}$, then $\sum_{r=0}^{7} C_{4r} =$

(a) 2^{28} (b) 2^{29} (c) 2^{30} (d) 2^{15}

JEE ADVANCED

6. Let $\vec{a}=2\hat{i}-\hat{j}+\hat{k}$, $\vec{b}=\hat{i}+2\hat{j}-\hat{k}$ and $\vec{c}=\hat{i}+\hat{j}-2\hat{k}$. A vector in the plane of $\vec{b}$ and $\vec{c}$ whose projection on $\vec{a}$ is of magnitude $\sqrt{\frac{2}{3}}$ is

(a) $2\hat{i}+3\hat{j}-3\hat{k}$ (b) $2\hat{i}+3\hat{j}+3\hat{k}$

(c) $-2\hat{i}-\hat{j}+5\hat{k}$ (d) $2\hat{i}+\hat{j}+5\hat{k}$

COMPREHENSION

Consider the odd integers 1, 3, 5, 7, 9,....

7. If $S = 1^3 - 3^3 + 5^3 - 7^3 + ... + 21^3$, then S is divisible by

(a) 5 (b) 7 (c) 11 (d) 13

8. If $S = 1^4 - 3^4 + 5^4 - 7^4 + ... + 25^4$, then the sum of digits of S is

(a) 15 (b) 20

(c) 25 (d) 30

INTEGER MATCH

9. If $2^{x+y} = 6^y$ and $2^x = 3 \times 2^{y+1}$, then $x = \log_a 3 \cdot \log_b 6$, where ab is

MATCHING LIST

10. Column II contains curves satisfying the conditions in Column I.

Column-I		Column-II	
P.	Subtangent is constant	1.	parabola
Q.	Subnormal is constant	2.	ellipse
R.	Subtangent = twice the abscissa	3.	hyperbola
S.	Subnormal = twice the abscissa	4.	exponential curve

	P	Q	R	S
(a)	4	1	1	3
(b)	2	3	3	1
(c)	1	4	4	2
(d)	1	2	3	4

See Solution set of Maths Musing 151 on page no. 80

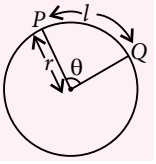
Prof. Ramanaiah is the author of MTG JEE(Main & Advanced) Mathematics series

MEASUREMENT OF AN ANGLE

- Sexagesimal or British system
- Centesimal or french system
- Circular system

1. **Sexagesimal or British System :** In this system of measurement, a right angle is divided into 90 equal parts called degrees.
   Each degree is then divided into 60 equal parts called minutes and each minute is further divided into 60 equal parts called seconds.
   Thus, 1 right angle = 90°, 1° = 60′ and 1′ = 60″
2. **Centesimal or French System:** In this system of measurement, a right angle is divided into 100 equal parts called grades. Each grade is then divided into 100 equal parts called minutes and each minute is further divided into 100 equal parts called seconds.
   Thus, 1 right angle = 100^g, $1^g = 100'$ and $1' = 100''$
3. **Circular System :** The angle subtended at the centre of a circle by an arc whose length is equal to the radius of the circle is called a radian and is denoted by 1^c.
   1 right angle = $(\pi/2)$ radians.

The length l of an arc PQ of a circle of radius r is given by $l = r\theta$, where θ is the angle subtended by arc PQ at the centre of the circle in terms of radians.



RELATION BETWEEN DEGREE AND RADIAN

Radian measure of an angle $= \frac{\pi}{180} \times$ degree measure of the angle

Degree measure of an angle $= \frac{180}{\pi} \times$ radian measure of the angle.

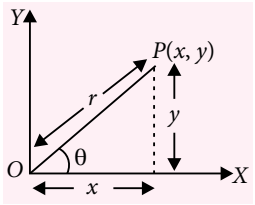
Thus, if the measure of an angle in degrees and radians be D and C respectively, then $\frac{D}{180} = \frac{C}{\pi}$

TRIGONOMETRIC FUNCTIONS

Trigonometric ratios for an acute angle is defined as the ratio of sides of a right angled triangle. The extension of trigonometric ratios to any angle in terms of radian measure are called trigonometric functions.

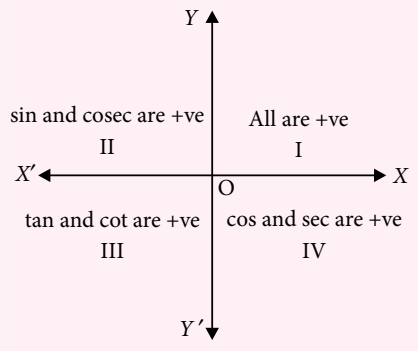
$\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$, $\tan\theta = \frac{y}{x}$,

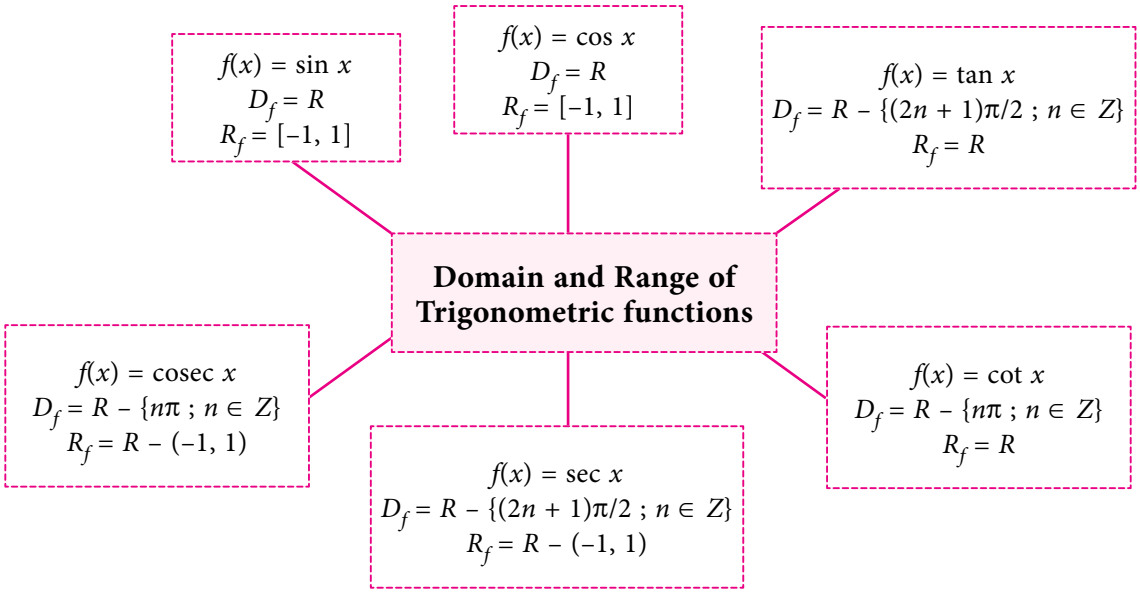
$\text{cosec}\,\theta = \frac{r}{y}$, $\sec\theta = \frac{r}{x}$, $\cot\theta = \frac{x}{y}$



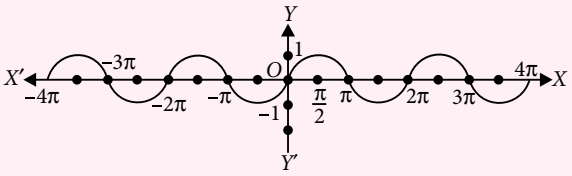
SIGNS OF TRIGONOMETRIC FUNCTIONS IN FOUR DIFFERENT QUADRANTS

Quadrant	Signs of various T-functions
I	All T-functions are positive.
II	sinθ and cosecθ are positive. All others are negative.
III	tanθ and cotθ are positive. All others are negative.
IV	cosθ and secθ are positve. All others are negative.

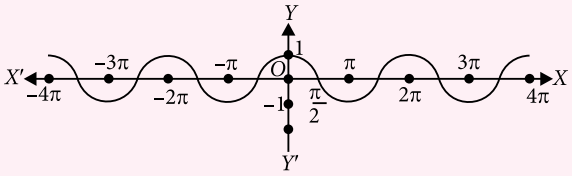



Domain and Range of Trigonometric functions
$f(x) = \sin x$
$D_f = R$
$R_f = [-1, 1]$
$f(x) = \cos x$
$D_f = R$
$R_f = [-1, 1]$
$f(x) = \tan x$
$D_f = R - \{(2n + 1)\pi/2 ; n \in Z\}$
$R_f = R$
$f(x) = \text{cosec } x$
$D_f = R - \{n\pi ; n \in Z\}$
$R_f = R - (-1, 1)$
$f(x) = \sec x$
$D_f = R - \{(2n + 1)\pi/2 ; n \in Z\}$
$R_f = R - (-1, 1)$
$f(x) = \cot x$
$D_f = R - \{n\pi ; n \in Z\}$
$R_f = R$

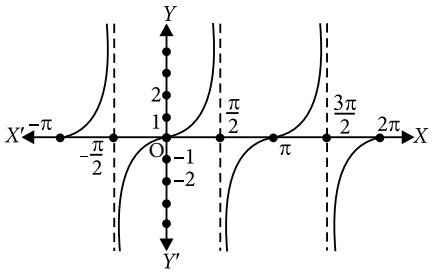
GRAPHS OF TRIGONOMETRIC FUNCTIONS



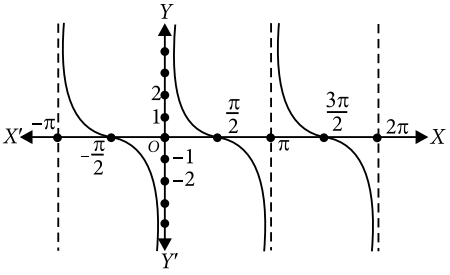
$y = \sin x$



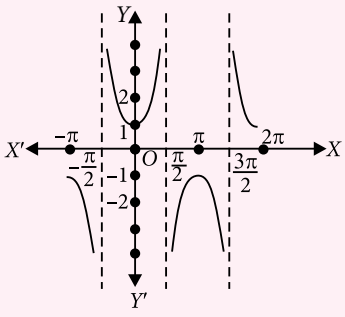
$y = \cos x$



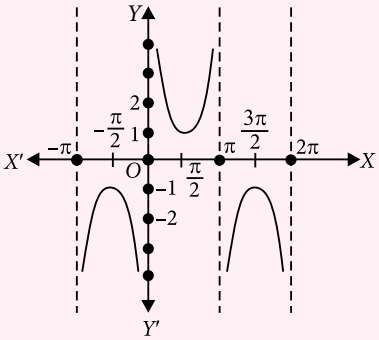
$y = \tan x$



$y = \cot x$



$y = \sec x$



$y = \text{cosec } x$

VALUES OF T-FUNCTIONS OF SOME PARTICULAR ANGLES

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	–1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	–1	0	1
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined	0	not defined	0

Note :

- $\sin(\theta + 2n\pi) = \sin\theta$, $\cos(\theta + 2n\pi) = \cos\theta$
  $\tan(\theta + n\pi) = \tan\theta$, $\cot(\theta + n\pi) = \cot\theta$
  $\sec(\theta + 2n\pi) = \sec\theta$, $\operatorname{cosec}(\theta + 2n\pi) = \operatorname{cosec}\theta$
- In case of $-\theta$, $\pi \pm \theta$, $2\pi \pm \theta$, t-ratio remains the same and in case of $\frac{\pi}{2} \pm \theta, \frac{3\pi}{2} \pm \theta$, t-ratio changes as follows : sin $\to$ cos, cos $\to$ sin, tan $\to$ cot, cot $\to$ tan, sec $\to$ cosec, cosec $\to$ sec.
- $\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$
  $\tan(-\theta) = -\tan\theta$, $\cot(-\theta) = -\cot\theta$
  $\sec(-\theta) = \sec\theta$, $\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta$

SOME IMPORTANT FORMULAE

- $\sin(A + B) = \sin A\cos B + \cos A\sin B$
- $\sin(A - B) = \sin A\cos B - \cos A\sin B$
- $\cos(A + B) = \cos A\cos B - \sin A\sin B$
- $\cos(A - B) = \cos A\cos B + \sin A\sin B$
- $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$
- $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$
- $\cot(A+B) = \dfrac{\cot A\cot B - 1}{\cot A + \cot B}$
- $\cot(A-B) = \dfrac{\cot A\cot B + 1}{\cot B - \cot A}$
- $\sin 2A = 2\sin A\cos A = \dfrac{2\tan A}{1+\tan^2 A}$
- $\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A$
  $= 2\cos^2 A - 1 = \dfrac{1-\tan^2 A}{1+\tan^2 A}$
- $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$
- $\sin 3A = 3\sin A - 4\sin^3 A$
- $\cos 3A = 4\cos^3 A - 3\cos A$
- $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$
- $\sin A + \sin B = 2\sin\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)$
- $\sin A - \sin B = 2\cos\left(\dfrac{A+B}{2}\right)\sin\left(\dfrac{A-B}{2}\right)$
- $\cos A + \cos B = 2\cos\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)$
- $\cos A - \cos B = 2\sin\left(\dfrac{A+B}{2}\right)\sin\left(\dfrac{B-A}{2}\right)$
- $2\sin A\cos B = \sin(A + B) + \sin(A - B)$
- $2\cos A\sin B = \sin(A + B) - \sin(A - B)$
- $2\cos A\cos B = \cos(A + B) + \cos(A - B)$
- $2\sin A\sin B = \cos(A - B) - \cos(A + B)$
- $\sin\dfrac{A}{2} = \pm\sqrt{\dfrac{1-\cos A}{2}}$ $\begin{cases} +\text{ ve}, + \text{if } \dfrac{A}{2} \text{ lies in I or II quadrants} \\ -\text{ve, if } \dfrac{A}{2} \text{ lies in III or IV quadrants} \end{cases}$
- $\cos\dfrac{A}{2} = \pm\sqrt{\dfrac{1+\cos A}{2}}$ $\begin{cases} +\text{ve, if } \dfrac{A}{2} \text{ lies in I or IV quadrants} \\ -\text{ve, if } \dfrac{A}{2} \text{ lies in II or III quadrants} \end{cases}$

- $\tan\frac{A}{2}=\pm\sqrt{\frac{1-\cos A}{1+\cos A}}$ $\begin{cases}+\text{ve, if } \frac{A}{2} \text{ lies in I or III quadrants}\\ -\text{ve, if } \frac{A}{2} \text{ lies in II or IV quadrants}\end{cases}$

TRIGONOMETRIC EQUATIONS

1. **Trigonometric equation :** Equation involving one or more trigonometric functions (ratios) of unknown angle is called a trigonometric equation.
2. **Solution or root of a trigonometric equation :** A value of the unknown angle which satisfies the given trigonometric equation is called a solution or root of the equation.
3. **Principal solution of a trigonometric equation :** The solutions of a trigonometric equation lying in the interval $[0, 2\pi)$ are called principal solutions.
4. **General solutions of some trigonometric equations:** The solution consisting of all possible solutions of a trigonometric equation is called its general solution.

- $\sin\theta = 0 \Rightarrow \theta = n\pi,\ n \in Z$
- $\cos\theta = 0 \Rightarrow \theta = (2n+1)\frac{\pi}{2},\ n \in Z$
- $\tan\theta = 0 \Rightarrow \theta = n\pi,\ n \in Z$
- $\cot\theta = 0 \Rightarrow \theta = (2n+1)\frac{\pi}{2},\ n \in Z$
- $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha,\ n \in Z$
- $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha,\ n \in Z$
- $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha,\ n \in Z$

PRINCIPLE OF MATHEMATICAL INDUCTION

Mathematical statements : Statements involving mathematical relations are known as the mathematical statements. We shall be using notation $P(n)$ or $P_1(n)$ or $P_2(n)$ etc. to denote such statements.

First principle of mathematical induction :

Let $P(n)$ be a statement involving the natural number n such that

(I) $P(1)$ is true *i.e.* $P(n)$ is true for $n = 1$ and

(II) $P(m + 1)$ is true, whenever $P(m)$ is true *i.e.* $P(m)$ is true $\Rightarrow P(m + 1)$ is true.

Then, $P(n)$ is true for all natural numbers n.

Second principle of mathematical induction :

Let $P(n)$ be a statement involving the natural number n such that

(I) $P(1)$ is true *i.e.* $P(n)$ is true for n =1 and

(II) $P(m + 1)$ is true, whenever $P(n)$ is true for all n, where $1 \le n \le m$.

Then, $P(n)$ is true for all natural numbers.

VERY SHORT ANSWER TYPE

1. Find the radian measure of $-37°30'$.
2. Find the value of cosec $(-1410°)$.
3. Prove that : $\frac{\sin 2x}{1-\cos 2x} = \cot x$
4. Find the general solution of cosec $\theta = 2$.
5. If $P(n)$ is the statement "$n(n + 1)\ (n + 2)$ is divisible by 12", prove that the statements $P(3)$ and $P(4)$ are true, but $P(5)$ is not true.

SHORT ANSWER TYPE

6. If $\tan\alpha = x + 1$, $\tan\beta = x - 1$, then show that $2\cot(\alpha - \beta) = x^2$.
7. If $\alpha + \beta = 90°$, find the maximum value of $\sin\alpha \sin\beta$.
8. Solve the equation $\sin x + \sin 3x + \sin 5x = 0$.
9. Using the principle of mathematical induction, prove that $n < 2^n$ for all $n \in N$.
10. If $\sin\theta = \frac{-4}{5}$ and $\pi < \theta < \frac{3\pi}{2}$, find the values of all the other five trigonometric functions.

LONG ANSWER TYPE

11. (i) Solve : $2\sin^2 x - 5\sin x\cos x - 8\cos^2 x = -2$

    (ii) Show that $\cot\left(\frac{\pi}{4}+x\right)\cdot\cot\left(\frac{\pi}{4}-x\right)=1$
12. Prove that $\cos^4\frac{\pi}{8}+\cos^4\frac{3\pi}{8}+\cos^4\frac{5\pi}{8}+\cos^4\frac{7\pi}{8}=\frac{3}{2}$,
13. If $\frac{\sin(\theta+\alpha)}{\cos(\theta-\alpha)}=\frac{1-m}{1+m}$, then prove that

    $\tan\left(\frac{\pi}{4}-\theta\right)\cdot\tan\left(\frac{\pi}{4}-\alpha\right)=m$
14. If $\frac{\tan(\theta+\alpha)}{a}=\frac{\tan(\theta+\beta)}{b}=\frac{\tan(\theta+\gamma)}{c}$, prove that

    $\frac{a+b}{a-b}\sin^2(\alpha-\beta)+\frac{b+c}{b-c}\sin^2(\beta-\gamma)+\frac{c+a}{c-a}\sin^2(\gamma-\alpha)=0$
15. If $A + B + C = \pi$, then prove that

    $\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}=2\left(1+\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$.

SOLUTIONS

1. we know that, $180^\circ = \pi^c \Rightarrow 1^\circ = \left(\frac{\pi}{180}\right)^c$

Now, $-37^\circ 30' = \left\{\frac{\pi}{180} \times \left(\frac{-75}{2}\right)\right\}^c = \left(\frac{-5\pi}{24}\right)^c$

2. $\text{cosec}(-1410^\circ) = -\text{cosec } 1410^\circ$

$= -\text{cosec}(4 \times 360^\circ - 30^\circ) = \text{cosec } 30^\circ = 2$

3. $\text{L.H.S.} = \frac{\sin 2x}{1-\cos 2x} = \frac{2\sin x \cos x}{2\sin^2 x} = \cot x = \text{R.H.S.}$

4. We have, $\text{cosec}\,\theta = 2$

$\Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \sin\theta = \sin\frac{\pi}{6}$

$\Rightarrow \theta = n\pi + (-1)^n \frac{\pi}{6}, n \in Z$

5. $P(3)$ is the statement "$3(3+1)(3+2) = 60$ is divisible by 12", which is true.

$P(4)$ is the statement "$4(4+1)(4+2) = 120$ is divisible by 12", which is also true.

$P(5)$ is the statement "$5(5+1)(5+2) = 210$ is divisible by 12". Clearly it is not true.

6. We have, $\tan\alpha = x + 1 \Rightarrow \cot\alpha = \frac{1}{x+1}$ ... (i)

$\tan\beta = x - 1 \Rightarrow \cot\beta = \frac{1}{x-1}$ ... (ii)

$$\text{L.H.S.} = 2\cot(\alpha-\beta) = \frac{2(\cot\alpha \cdot \cot\beta + 1)}{\cot\beta - \cot\alpha}$$

$$= \frac{2\left(\frac{1}{x+1}\cdot\frac{1}{x-1}+1\right)}{\left(\frac{1}{x-1}-\frac{1}{x+1}\right)} \quad \text{[From (i) and (ii)]}$$

$$= \frac{2\left(\frac{1}{x^2-1}+1\right)}{\frac{(x+1)-(x-1)}{x^2-1}} = 2\cdot\frac{x^2}{2} = x^2 = \text{R.H.S}$$

7. Let $y = \sin\alpha \sin\beta$

$\Rightarrow y = \frac{1}{2}(2\sin\alpha\sin\beta) = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$

$= \frac{1}{2}[\cos(\alpha-\beta) - \cos 90^\circ] = \frac{1}{2}\cos(\alpha-\beta)$

Since maximum value of $\cos(\alpha - \beta)$ is 1

$\therefore$ Maximum value of $y = \frac{1}{2}$

8. Given equation is $\sin x + \sin 3x + \sin 5x = 0$

$\Rightarrow (\sin 5x + \sin x) + \sin 3x = 0$

$\Rightarrow 2\sin 3x \cos 2x + \sin 3x = 0$

$\Rightarrow \sin 3x (2\cos 2x + 1) = 0$

$\Rightarrow \sin 3x = 0$ or $2\cos 2x + 1 = 0$

Now $\sin 3x = 0 \Rightarrow 3x = n\pi, n \in Z$

$\Rightarrow x = \frac{n\pi}{3}, n \in Z$ ...(i)

Also, $2\cos 2x + 1 = 0$

$\Rightarrow \cos 2x = -\frac{1}{2} \Rightarrow \cos 2x = \cos\frac{2\pi}{3}$

$\Rightarrow 2x = 2n\pi \pm \frac{2\pi}{3}, n \in Z$

$\Rightarrow x = (3n \pm 1)\frac{\pi}{3}, n \in Z$ ...(ii)

But solution in (ii) are induced in solution (i), because $(3n \pm 1)$ is also an integer. Hence, required solution is $x = \frac{n\pi}{3}, n \in Z$

9. Let $P(n) : n < 2^n \ \forall n \in N$

When $n = 1$, L.H.S. $= 1$ and R.H.S. $= 2^1 = 2$

Clearly, $1 < 2$

Thus, $P(n)$ is true for $n = 1$, *i.e.*, $P(1)$ is true.

Let $P(k)$ be true. Then, $P(k) : k < 2^k \ \forall\ k \in N$

Now, $k < 2^k$

$\Rightarrow 2k < 2^{k+1} \Rightarrow (k+k) < 2^{k+1}$

$\Rightarrow (k+1) \le (k+k) < 2^{k+1}$ $[\because\ 1 \le k]$

$\Rightarrow (k+1) < 2^{k+1}$

$\therefore\ P(k+1) : (k+1) < 2^{k+1}$

$\Rightarrow P(k+1)$ is true, whenever $P(k)$ is true.

Thus, $P(1)$ is true and $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in N$.

10. Clearly, θ lies in the third quadrant in which $\tan\theta$ and $\cot\theta$ are positive and all the other trigonometric functions are negative.

We have, $\sin\theta = \frac{-4}{5}, \text{cosec}\,\theta = \frac{1}{\sin\theta} = \frac{-5}{4}$

Now, $\cot^2\theta = (\text{cosec}^2\theta - 1) = \left(\frac{25}{16} - 1\right) = \frac{9}{16}$

$\Rightarrow \cot\theta = \sqrt{\frac{9}{16}} = \frac{3}{4}, \ \tan\theta = \frac{1}{\cot\theta} = \frac{4}{3}$

Also, $\cos\theta = -\sqrt{1-\sin^2\theta} = -\sqrt{1-\frac{16}{25}} = -\frac{3}{5}$

$\therefore \sec\theta = \frac{1}{\cos\theta} = \frac{-5}{3}$

Hence, $\cos\theta = \frac{-3}{5}$; $\tan\theta = \frac{4}{3}$; $\cot\theta = \frac{3}{4}$;

$\sec\theta = \frac{-5}{3}$ and $\operatorname{cosec}\theta = \frac{-5}{4}$

11. (i) Given equation is

$2\sin^2 x - 5\sin x\cos x - 8\cos^2 x = -2$

Dividing both sides by $\cos^2 x$, we get

$2\tan^2 x - 5\tan x - 8 = -2\sec^2 x$

$\Rightarrow 2\tan^2 x - 5\tan x - 8 + 2(1 + \tan^2 x) = 0$

$\Rightarrow 4\tan^2 x - 5\tan x - 6 = 0$

$\Rightarrow 4\tan^2 x - 8\tan x + 3\tan x - 6 = 0$

$\Rightarrow 4\tan x(\tan x - 2) + 3(\tan x - 2) = 0$

$\Rightarrow (\tan x - 2)(4\tan x + 3) = 0$

$\therefore$ Either $\tan x - 2 = 0 \Rightarrow \tan x = 2 = \tan\alpha$ (say)

$\Rightarrow x = n\pi + \alpha$, where $\tan\alpha = 2$, $n \in Z$

or $4\tan x + 3 = 0 \Rightarrow \tan x = -\frac{3}{4} = \tan\beta$ (say)

$\Rightarrow x = n\pi + \beta$, where $\tan\beta = -\frac{3}{4}$, $n \in Z$

(ii) $\text{L.H.S.} = \dfrac{\cos\left(\frac{\pi}{4}+x\right)\cdot\cos\left(\frac{\pi}{4}-x\right)}{\sin\left(\frac{\pi}{4}+x\right)\sin\left(\frac{\pi}{4}-x\right)}$

$= \dfrac{\cos^2\frac{\pi}{4} - \sin^2 x}{\sin^2\frac{\pi}{4} - \sin^2 x} = \dfrac{\frac{1}{2} - \sin^2 x}{\frac{1}{2} - \sin^2 x} = 1$

12. We have, $\frac{7\pi}{8} = \pi - \frac{\pi}{8}$ and $\frac{5\pi}{8} = \pi - \frac{3\pi}{8}$

$\Rightarrow \cos\frac{7\pi}{8} = -\cos\frac{\pi}{8}$ and $\cos\frac{5\pi}{8} = -\cos\frac{3\pi}{8}$

$\Rightarrow \cos^4\frac{7\pi}{8} = \cos^4\frac{\pi}{8}$ and $\cos^4\frac{5\pi}{8} = \cos^4\frac{3\pi}{8}$

$\therefore$ L.H.S $= 2\cos^4\frac{\pi}{8} + 2\cos^4\frac{3\pi}{8}$

$= 2\left[\left(\cos^2\frac{\pi}{8}\right)^2 + \left(\cos^2\frac{3\pi}{8}\right)^2\right]$

$= 2\left[\left\{\frac{1+\cos\frac{\pi}{4}}{2}\right\}^2 + \left\{\frac{1+\cos\frac{3\pi}{4}}{2}\right\}^2\right]$

$= \frac{2}{4}\left\{\left(1+\cos\frac{\pi}{4}\right)^2 + \left(1+\cos\frac{3\pi}{4}\right)^2\right\}$

$= \frac{1}{2}\left\{\left(1+\frac{1}{\sqrt{2}}\right)^2 + \left(1-\frac{1}{\sqrt{2}}\right)^2\right\} = \frac{3}{2} = \text{RHS}$

13. Given, $\dfrac{\sin(\theta+\alpha)}{\cos(\theta-\alpha)} = \dfrac{1-m}{1+m}$

$\Rightarrow \dfrac{\sin(\theta+\alpha)+\cos(\theta-\alpha)}{\sin(\theta+\alpha)-\cos(\theta-\alpha)} = \dfrac{1-m+1+m}{1-m-1-m}$

[By componendo and dividendo rule]

$\Rightarrow \dfrac{\sin(\theta+\alpha)+\sin\left[\frac{\pi}{2}-(\theta-\alpha)\right]}{\sin(\theta+\alpha)-\sin\left[\frac{\pi}{2}-(\theta-\alpha)\right]} = \dfrac{2}{-2m} = -\dfrac{1}{m}$

$\Rightarrow \dfrac{2\sin\frac{\theta+\alpha+\frac{\pi}{2}-\theta+\alpha}{2}\cos\frac{\theta+\alpha-\frac{\pi}{2}+\theta-\alpha}{2}}{2\cos\frac{\theta+\alpha+\frac{\pi}{2}-\theta+\alpha}{2}\sin\frac{\theta+\alpha-\frac{\pi}{2}+\theta-\alpha}{2}} = -\dfrac{1}{m}$

$\Rightarrow \dfrac{\sin\left(\frac{\pi}{4}+\alpha\right)\cdot\cos\left(\theta-\frac{\pi}{4}\right)}{\cos\left(\frac{\pi}{4}+\alpha\right)\cdot\sin\left(\theta-\frac{\pi}{4}\right)} = -\dfrac{1}{m}$

$\Rightarrow \tan\left(\frac{\pi}{4}+\alpha\right)\cdot\cot\left(\theta-\frac{\pi}{4}\right) = -\frac{1}{m}$

$\Rightarrow -\tan\left(\frac{\pi}{4}+\alpha\right)\cot\left(\frac{\pi}{4}-\theta\right) = -\frac{1}{m}$

$\Rightarrow m = \cot\left(\frac{\pi}{4}+\alpha\right)\tan\left(\frac{\pi}{4}-\theta\right)$

$= \tan\left[\frac{\pi}{2} - \left(\frac{\pi}{4}+\alpha\right)\right]\tan\left(\frac{\pi}{4}-\theta\right)$

$= \tan\left(\frac{\pi}{4}-\theta\right)\cdot\tan\left(\frac{\pi}{4}-\alpha\right)$

14. We have, $\dfrac{\tan(\theta+\alpha)}{a} = \dfrac{\tan(\theta+\beta)}{b}$

$\Rightarrow \dfrac{a}{b} = \dfrac{\tan(\theta+\alpha)}{\tan(\theta+\beta)}$

$\Rightarrow \dfrac{a+b}{a-b} = \dfrac{\tan(\theta+\alpha)+\tan(\theta+\beta)}{\tan(\theta+\alpha)-\tan(\theta+\beta)}$

$\Rightarrow \dfrac{a+b}{a-b} = \dfrac{\sin(2\theta+\alpha+\beta)}{\sin(\alpha-\beta)}$

$$\Rightarrow \frac{a+b}{a-b}\sin^2(\alpha-\beta)=\sin(2\theta+\alpha+\beta)\sin(\alpha-\beta)$$

$$\Rightarrow \frac{a+b}{a-b}\sin^2(\alpha-\beta)=\frac{1}{2}\{2\sin(2\theta+\alpha+\beta)\sin(\alpha-\beta)\}$$

$$\Rightarrow \frac{a+b}{a-b}\sin^2(\alpha-\beta)=\frac{1}{2}\{\cos(2\theta+2\beta)-\cos(2\theta+2\alpha)\} \quad \text{... (i)}$$

Similarly, we have

$$\frac{b+c}{b-c}\sin^2(\beta-\gamma)=\frac{1}{2}\{\cos(2\theta+2\gamma)-\cos(2\theta+2\beta)\} \quad \text{... (ii)}$$

and, $$\frac{c+a}{c-a}\sin^2(\gamma-\alpha)=\frac{1}{2}\{\cos(2\theta+2\alpha)-\cos(2\theta+2\gamma)\} \quad \text{... (iii)}$$

Adding (i), (ii) and (iii), we get

$$\frac{a+b}{a-b}\sin^2(\alpha-\beta)+\frac{b+c}{b-c}\sin^2(\beta-\gamma)+\frac{c+a}{c-a}\sin^2(\gamma-\alpha)$$

$$=\frac{1}{2}\{\cos(2\theta+2\beta)-\cos(2\theta+2\alpha)+\cos(2\theta+2\gamma)-\cos(2\theta+2\beta)+\cos(2\theta+2\alpha)-\cos(2\theta+2\gamma)\}$$

$$=\frac{1}{2}\times 0=0$$

15. L.H.S $= \cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}$

$$=\frac{1}{2}(1+\cos A)+\frac{1}{2}(1+\cos B)+\frac{1}{2}(1+\cos C)$$

$$=\frac{3}{2}+\frac{1}{2}(\cos A+\cos B+\cos C)$$

$$=\frac{3}{2}+\frac{1}{2}\left[2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)+\cos C\right]$$

$$=\frac{3}{2}+\cos\left(\frac{\pi}{2}-\frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right)+\frac{1}{2}\left(1-2\sin^2\frac{C}{2}\right)$$

$$=2+\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right)-\sin^2\frac{C}{2}$$

$$=2+\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right)-\sin\frac{C}{2}\right]$$

$$=2+\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right)-\sin\left\{\frac{\pi}{2}-\left(\frac{A+B}{2}\right)\right\}\right]$$

$$=2+\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{A+B}{2}\right)\right]$$

$$=2+\sin\frac{C}{2}\left(2\sin\frac{A}{2}\sin\frac{B}{2}\right)$$

$$=2\left(1+\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)=\text{R.H.S.}$$

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Algebra Checkouts

1. It is given that the polynomial $P(x) = x^3 + ax^2 + bx + c$ has three distinct positive integer roots and $P(14) = 231$. Let $Q(x) = x^2 - 2x + 232$. It is also given that the polynomial $P(Q(x))$ has no real zeros. Then $|a| =$

(a) 21 (b) 43 (c) 67 (d) 91

2. Evaluate the infinite series

$$S = 1 - \frac{2^3}{1!} + \frac{3^3}{2!} - \frac{4^3}{3!} + \ldots.$$

(a) $\frac{1}{e}$ (b) $\frac{-1}{e}$ (c) $\frac{2}{e}$ (d) $\frac{-2}{e}$

3. The smallest possible sum of $a_1b_1 + a_2b_2 + \ldots + a_8b_8$ where $a_1, a_2, \ldots, a_8$ and $b_1, b_2, \ldots, b_8$ are rearrangements of the binomial coefficients $({}^7C_0, {}^7C_1, {}^7C_2, \ldots, {}^7C_7)$ is

(a) ${}^{14}C_3$ (b) $2 \cdot {}^{14}C_3$

(c) $\frac{1}{2} \cdot {}^{14}C_7$ (d) $({}^{14}C_7)^2$

4. In the sequence 1, 2, 2, 4, 4, 4, 4, 8, 8, 8, 8, 8, 8, 8, 8, ..., the number occupying the 100^{th} position is

(a) 2^6 (b) 2^{10} (c) 2^{100} (d) 2^{101}

5. Evaluate

$$\frac{a_1^3}{1-3a_1+3a_1^2} + \frac{a_2^3}{1-3a_2+3a_2^2} + \ldots + \frac{a_{101}^3}{1-3a_{101}+3a_{101}^2}$$

where each $a_n = \frac{n}{101}$

(a) 50 (b) 51 (c) 100 (d) 102

6. The number of positive real numbers x satisfying the equation $(8^x - 5^x)(7^x - 2^x)(6^x - 4^x) + (9^x - 4^x)(8^x - 3^x)(5^x - 2^x) = (105)^x$ is k then $\left[\frac{k}{10}\right] =$, ([·] denotes G.I.F.)

(a) 0 (b) 1 (c) 2 (d) 3

7. A gambler tosses a fair coin and scores one point for each head that turns up and two points for each tail. The probability of the gambler scoring exactly 10 points is

(a) $\frac{2049}{3072}$ (b) $\frac{3012}{3048}$ (c) $\frac{873}{1024}$ (d) $\frac{419}{1512}$

8. If three distinct numbers a, b, c are roots of the equation $x^3 - x^2 - x - 1 = 0$ then a possible value of $\frac{a^\alpha - b^\alpha}{a-b} + \frac{b^\alpha - c^\alpha}{b-c} + \frac{c^\alpha - a^\alpha}{c-a}$ $(\alpha > 21, \alpha \in N)$ is

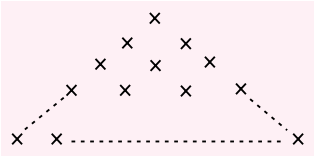
(a) $\sqrt{2}$ (b) 2 (c) $\frac{3}{10}$ (d) $-\frac{1}{3}$

9. Let set $S = \{a_1, a_2, \ldots, a_n\}$. The G.M. of the G.M.'s of all non-empty subsets of S is

(a) $\left(\prod_{i=1}^{n} a_i\right)^{1/n}$ (b) $\prod_{i=1}^{n} a_i$

(c) $\frac{1}{n}\left(\prod_{i=1}^{n} a_i\right)$ (d) $\frac{1}{n}\left(\sum_{i=1}^{n} a_i\right)$

10. A set of $\frac{n(n+1)}{2}$ distinct numbers are arranged at random in a triangular arrangement as shown. Let T_k be the largest number in the k^{th} row from the top. The probability that $T_1 < T_2 < T_3 < \ldots < T_n$ is $(n \geq 3)$



(a) $\frac{2^n}{(n+1)!}$ (b) $\frac{2^{n+1}}{(n+1)!}$ (c) $\frac{n+1}{2^n}$ (d) $\frac{n}{2^{n+1}}$

11. Two persons Mr. A and Mr. B randomly select an integer from 1 to 9, independently of each other. The most likely value(s) for the units digits of the sum of their numbers is

(a) 0 (b) 3 (c) 7 (d) 9

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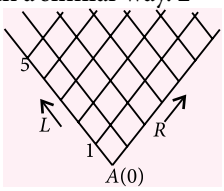
12. The reduced fraction $\frac{p}{q}$ with $q \in [8, 99]$ is closest to $\frac{3}{7}$, then $q =$

(a) 23 (b) 45 (c) 83 (d) 96

13. The number of real values of p for which the solutions α, β of the equation $x^2 - 2px + p^2 - 2p - 1 = 0$ have the property that $\frac{(\alpha-\beta)^2-2}{2[(\alpha+\beta)^2+2]}$ is an integer is/are

(a) zero (b) 2 (c) 4 (d) ∞

14. The bottom part of a network of roads is shown below. It extends upto level 99 in a similar way. 2^{100} people leave point A (level 0). At every intersection half go in L direction and half in R direction. The number of people ending up at the next-to-left position at level 99 is



(a) 100 (b) 198 (c) 2^{10} (d) 2^9

15. Suppose $a_{n+1} = 2(a_n + b_n)$, $b_{n+1} = 2b_n$ for all $n \in N$. If $a_{n+3} = ta_n + kb_n$ and $b_{n+3} = ya_n + gb_n$ then $t + k + y + g =$

(a) 40 (b) 45 (c) -40 (d) -45

16. The value of $4\sqrt{3} + \sqrt{129 - 72\sqrt{3}} =$

(a) 9 (b) $9\sqrt{3}$

(c) $9 + \sqrt{3}$ (d) $9 - \sqrt{3}$

17. A set of positive integers has the Y-property if it has three distinct elements that are the lengths of the sides of a triangle whose area is positive. Consider the set $\{4, 5, 6, ..., t\}$ of consecutive integers, all of whose 10 element subsets have the Y-property. The largest possible value of t is

(a) 251 (b) 252 (c) 253 (d) 254

18. The number of integers for which $\frac{8(n^3 - 3n^2 + 4)}{2n - 1}$ is an integer is

(a) 4 (b) 8 (c) 11 (d) 28

19. Let $E = 1!\ 2!\ 3!\ ...\ 100!$. The factor that must be erased from E so that E becomes a perfect square is

(a) 25! (b) 50! (c) 99! (d) 100!

20. Mr. X wrote 21 numbers on a circle. Each number was half the sum of its two neighbours. If one of the number is 3, the sum of all the numbers written on the circle is

(a) 21 (b) 63 (c) 42 (d) 84

21. The number of common terms in the two equations

A : 1, 1, 3, 5, 11, 21, ...

B : 1, 7, 17, 55, 161, 487, ... is/are

(a) 1 (b) 2 (c) 10 (d) 21

22. A natural number n for which $3^9 + 3^{12} + 3^{15} + 3^n$ is a perfect cube of an integer is

(a) 13 (b) 14 (c) 17 (d) 19

23. The largest 3-digit prime factor of the integer ${}^{2000}C_{1000}$ is

(a) 559 (b) 661 (c) 743 (d) 893

SOLUTIONS

1. (a) : $Q(x) = x^2 - 2x + 1 + 231 = (x - 1)^2 + 231$

$\Rightarrow$ $Q(x)$ takes all values from $[231, \infty)$

Let $P(x) = (x - x_1)(x - x_2)(x - x_3)$

$P(14) = 231 = (14 - x_1)(14 - x_2)(14 - x_3) = 3 \times 7 \times 11$

So, $14 - x_1 = 3$, $14 - x_2 = 7$, $14 - x_3 = 11$

$a = -(x_1 + x_2 + x_3) = -(11 + 7 + 3)$

$\Rightarrow$ $|a| = 21$

2. (b) : $(n+1)^{\text{th}}$ term $T_{n+1} = (-1)^n \cdot \frac{(n+1)^3}{n!}$, $n = 0$ to ∞

$$= \frac{(-1)^n}{n!}[n(n-1)(n-2) + 6n(n-1) + 7n + 1]$$

$$= \frac{(-1)^n}{(n-3)!} + \frac{6(-1)^n}{(n-2)!} + \frac{7(-1)^n}{(n-1)!} + \frac{(-1)^n}{n!}$$

$$S = \sum_{n=0}^{\infty} T_{n+1} = \frac{-1}{e}$$

3. (b) : By rearrangement inequality, we know that $\Sigma a_1 b_1$ is minimized when one of the two sequences $(a_1, a_2, ...)$ and $(b_1, b_2, ...)$ is increasing and other is decreasing. Since, the binomial coefficients are increasing from $({}^7C_0$ to ${}^7C_3)$ and decreasing from $({}^7C_4$ to ${}^7C_7)$, the minimum sum is $2\sum_{k=0}^{3} ({}^7C_k)({}^7C_{k+4})$

$$= 2\sum_{k=0}^{3} ({}^7C_k)({}^7C_{3-k}) = 2[{}^{14}C_3]$$

4. (a) : $a_n = 2^k$, where k is the largest power of 2 appearing in the expansion of n when n is expressed as sum of powers of 2.

$100 = 2^2 + 2^5 + 2^6$, so $a_{100} = 2^6$

5. (b) : $T_n = \frac{a_n^3}{1 - 3a_n + 3a_n^2} = \frac{a_n^3}{a_n^3 + (1 - a_n)^3}$

$$= \frac{n^3}{n^3 + (101 - n)^3}$$

$\Rightarrow$ $T_n + T_{101-n} = 1$

So $T_1 + T_{100} = 1$
$T_2 + T_{99} = 1$

$T_{50} + T_{51} = 1$
$\Rightarrow S_{100} = 50$
$S_{100} + T_{101} = 51$

6. (a) : Notice that $x = 1$ is a solution.
Dividing throughout by $(105)^x$, we have

$$\left[\left(\frac{8}{5}\right)^x - 1^x\right]\left[1^x - \left(\frac{2}{7}\right)^x\right]\left[2^x - \left(\frac{4}{3}\right)^x\right] + \left[\left(\frac{9}{7}\right)^x - \left(\frac{4}{7}\right)^x\right]\left[\left(\frac{8}{3}\right)^x - 1^x\right]\left[1^x - \left(\frac{2}{5}\right)^x\right] = 1$$

Now, using the concept that if $a > b > 0$, then $a^x - b^x$ is monotonically increasing. For $x \geq 0$, we see that the above equation has no solution other than $x = 1$.
So, only one solution exist.

7. (a) : Let P_n be the required probability then

$$P_n = \frac{1}{2}\cdot P_{n-1} + \frac{1}{2}P_{n-2} \text{ with } P_1 = \frac{1}{2} \text{ and}$$
$$P_2 = \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{3}{4}$$

Forming the character equation of the above equation and solving it, we have $P_n = \frac{1}{3}\left[2 + \left(-\frac{1}{2}\right)^n\right]$

and so, $P_{10} = \frac{2049}{3072}$

8. (b) : Let $r_\alpha = \frac{a^\alpha - b^\alpha}{a-b}$, $s_\alpha = \frac{b^\alpha - c^\alpha}{b-c}$, $t_\alpha = \frac{c^\alpha - a^\alpha}{c-a}$

Now, $(a^3 - a^2 - a - 1 = 0)\times a^n$
$(b^3 - b^2 - b - 1 = 0)\times b^n \downarrow (+)$
$r_{n+3} - r_{n+2} - r_{n+1} - r_n = 0$

Similarly, for s and t. Now an easy induction proves that $r_n + s_n + t_n$ is an integer for all $n \in N$.

9. (a) : Consider a particular element a_i. It occurs in ${}^{n-1}C_{k-1}$, k element subsets of S with exponent $\frac{1}{k}$ in each of the corresponding G.M.'s. Hence the exponent of a_i in the G.M. of the G.M.'s is given by

$$\frac{1}{2^n - 1}\cdot\sum_{k=1}^{n} {}^{n-1}C_{k-1}\cdot\frac{1}{k} = \frac{1}{2^n-1}\sum_{k=1}^{n}{}^nC_k\cdot\frac{1}{n}$$
$$= \frac{1}{n}\cdot\frac{1}{2^n-1}\cdot\sum_{k=1}^{n}{}^nC_k = \frac{1}{n}$$

which is same as that in the G.M. of S.

10. (a) : Let P_n be the probability when there are n rows. Clearly $P_1 = 1$, $P_2 = 2/3$. In general, the largest number should go in the last row, the probability for which is $\frac{n}{\frac{n(n+1)}{2}} = \frac{2}{n+1}$. There is no restriction on the remaining numbers of the last row. The probability that the numbers of the previous $(n-1)$ rows are suitably placed is P_{n-1}. Hence, $P_n = \frac{2}{n+1}\cdot P_{n-1}$

So, $P_n = \frac{2^n}{(n+1)!}$

11. (a) : Out of 81 equally likely pairs, lets find the number of ways each sum can occur.

2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	2	3	4	5	6	7	8	9	8	7	6	5	4	3	2	1

So, last digit 0 occurs in 9 ways and other last digits occurs in 8 ways.

12. (d) : Consider $\left|\frac{3}{7} - \frac{p}{q}\right| = \frac{|3q-7p|}{7q}$

we find q as large as possible with $|3q - 7p| = 1$. We look for multiples of 7 which are slightly less than 300 and are not divisible by 3. The largest is $287 = 41 \times 7$ which differs from 3×96 by 1. So reduced fraction $\frac{p}{q}$ is $\frac{41}{96}$.

13. (b) : $\alpha + \beta = 2p$, $\alpha\beta = p^2 - 2p - 1$
$\Rightarrow (\alpha - \beta)^2 = 8p + 4$

$$n = \frac{4p+1}{4p^2+2} \in I \Rightarrow p = \frac{-1}{4} \text{ and}$$
$$p = \frac{1 \pm \sqrt{1-n(2n-1)}}{2n}$$
$\Rightarrow n = 1$, so $p = \frac{1}{2}$.

14. (b) : This is like Pascal's triangle. At level n, the fraction of people in horizontal position k is $\frac{{}^nC_k}{2^n}$ and so the number of people in position at level 99 is $\frac{{}^{99}C_1}{2^{99}}\times 2^{100} = 198$

15. (a) : $\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = 2A\begin{pmatrix} a_n \\ b_n \end{pmatrix}$ where $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

So, $\begin{pmatrix} a_{n+3} \\ b_{n+3} \end{pmatrix} = 8A^3\begin{pmatrix} a_n \\ b_n \end{pmatrix}$
$\Rightarrow t = 8$, $k = 24$, $y = 0$, $g = 8$

16. (a) : $4\sqrt{3}+\sqrt{(9-4\sqrt{3})^2}=4\sqrt{3}+|9-4\sqrt{3}|=9$

17. (c) : Let us find a 10-element set S that does not have Y-property. Starting from the smallest possible values $S = \{4, 5, (4 + 5), 5 + (4 + 5), ...\}$
$S \equiv \{4, 5, 9, 14, 23, 37, 60, 97, 157, 254\}$.
So the largest set having the Y-property is
$\{4, 5, 6, ..., 253\}$
So, $t_{max} = 253$

18. (b) : Note that
$$\frac{8(n^3-3n^2+4)}{(2n-1)}=4n^2-10n-5+\frac{27}{2n-1}$$
So, $2n - 1 = \pm27, \pm9, \pm3, \pm1$

19. (b) : $E = 1!\ 2!\ 3!\ ...\ 100!$
$= (1!)^2 \cdot 2 \cdot (3!)^2 \cdot 4\ ....\ (99!)^2 \cdot 100$
$= (1!\ 3!\ 5!\ ...\ 99!)^2 \times 2^{50} \times [1 \cdot 2\ ...\ \cdot 50]$
$= (1!\ 3!\ 5!\ ...\ 99!)^2 \times 2^{50} \times 50!$

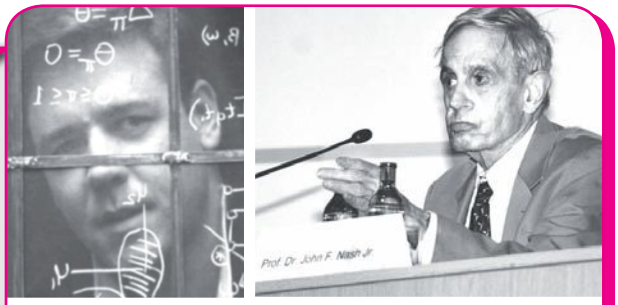
20. (b) : Let a_i be the maximum of the numbers written then $a_i \geq a_{i+1}$ and $a_i \geq a_{i-1}$
$$\Rightarrow\ a_i \geq \frac{a_{i+1}+a_{i-1}}{2}$$
but according to question, $a_i=\frac{a_{i+1}+a_{i-1}}{2}$
So, $a_i = a_{i+1}$ and $a_i = a_{i-1}$
$\Rightarrow$ All numbers are same = 21.
$\therefore$ Sum of numbers = $3 \times 21 = 63$.

21. (a) : mod 8 : A : 1, 1, 3, 5, 3, 5, ...
B : 1, 7, 1, 7, 1, 7, ...
An easy induction reveals that A and B are periodic and 1 is the only common term.

22. (b) : $(3^3)^3[1 + 3 \cdot 3^2 + 3^6 + 3^{n-9}]$
$\Rightarrow\ 3^{n-9} - 3^5 = 0$
So, $n = 14$

23. (b) : $^{2000}C_{1000}=\frac{2000\cdot1999\cdot....1001}{1000\cdot999\cdot.....3\cdot2\cdot1}$
The greatest 3-digit prime factor is the one which occurs once in denominator and atleast twice in numerator.
It must be less than $\frac{1}{3}\times2000$ and near to it.

A mathematical mind falls silent



John, Nash's decades-long descent into severe mental illness and eventual recovery were the subject of a book and a 2001 film, both titled A Beautiful Mind; (left) Russel Crowe in the Oscar-winning role.

John F. Nash Jr., a mathematician who shared a Nobel Prize in 1994 for work that greatly extended the reach and power of modern economic theory and whose decades-long descent into severe mental illness and eventual recovery were the subject of a book and a 2001 film, both titled *A Beautiful Mind*, was killed, along with his wife, in a car crash on Saturday in New Jersey. He was 86.

Mr. Nash was widely regarded as one of the great mathematicians of the 20th century, known for the originality of his thinking and for his fearlessness in wrestling down problems so difficult few others dared tackle them. A one-sentence letter written in support of his application to Princeton's doctoral program in math said simply, "This man is a genius."

"John's remarkable achievements inspired generations of mathematicians, economists and scientists," the president of Princeton, Christopher L. Eisgruber, said, "and the story of his life with Alicia moved millions of readers and moviegoers who marvelled at their courage in the face of daunting challenges."

Russell Crowe, who portrayed Nash in *A Beautiful Mind*, tweeted that he was "stunned," by his death. "An amazing partnership," he wrote. "Beautiful minds, beautiful hearts."

Mr. Nash's theory of non-cooperative games, published in 1950 and known as Nash equilibrium, provided a conceptually simple but powerful mathematical tool for analysing a wide range of competitive situations, from corporate rivalries to legislative decision making.

Mr. Nash's approach is now pervasive in economics and throughout the social sciences and is applied routinely in other fields, like evolutionary biology.

Among top 10 ideas

Harold W. Kuhn, an emeritus professor of mathematics at Princeton and a long-time friend and colleague of Mr. Nash's who died in 2014, said, "I think honestly that there have been really not that many great ideas in the 20th century in economics and maybe, among the top 10, his equilibrium would be among them."

> "Nash's theory of non-cooperative games provided a powerful mathematical tool for analysing a wide range of competitive situations"

An economist, Roger Myerson of the University of Chicago, went further, comparing the impact of Nash equilibrium on economics "to that of the discovery of the DNA double helix in the biological sciences."

Mr. Nash also made contributions to pure mathematics that many mathematicians view as more significant than his Nobel-winning work on game theory, including solving an intractable problem in differential geometry derived from the work of the 19th century mathematician G.F.B. Riemann.

His achievements were the more remarkable, colleagues said, for being contained in a small handful of papers published before he was 30.

Yet to a wider audience, Mr. Nash was probably best known for his life story, a tale of dazzling achievement, devastating loss and almost miraculous redemption.

The narrative of Mr. Nash's brilliant rise, the lost years when his world dissolved in schizophrenia, his return to rationality and the awarding of the Nobel, retold in a biography by Sylvia Nasar and in the Oscar-winning film, captured the public mind and became a symbol of the destructive force of mental illness and the stigma that often hounds those who suffer from it. — New York Times News Service

Courtesy : The Hindu

AMU ENGG.

SOLVED PAPER 2015

1. Let $A = [a_{ij}]_{m \times n}$ be a matrix such that $a_{ij} = 1$ for all i, j. Then
(a) Rank $(A) > 1$ (b) Rank $(A) = 1$
(c) Rank $(A) = m$ (d) Rank $(A) = n$

2. The equation
$\sqrt{(x-2)^2 + y^2} + \sqrt{(x+2)^2 + y^2} = 4$ represents
(a) a circle (b) a parabola
(c) a pair of lines (d) an ellipse

3. $\lim\limits_{n \to \infty} \dfrac{(n!)^{1/n}}{n}$ equals
(a) e (b) e^{-1} (c) 1 (d) None of these

4. Given that $\int_0^{\infty} \dfrac{x^2}{(x^2+a^2)(x^2+b^2)(x^2+c^2)}\,dx$
$= \dfrac{\pi}{2(a+b)(b+c)(c+a)}$, then $\int_0^{\infty} \dfrac{dx}{(x^2+4)(x^2+9)}$ is
(a) $\dfrac{\pi}{60}$ (b) $\dfrac{\pi}{20}$ (c) $\dfrac{\pi}{40}$ (d) $\dfrac{\pi}{80}$

5. For all real x, $4^{\sin^2 x} + 4^{\cos^2 x}$ is
(a) ≥ 4 (b) > 4
(c) < 4 (d) None of these

6. The length of the perpendicular drawn from $(1, 2, 3)$ to the line $\dfrac{x-6}{3} = \dfrac{y-7}{2} = \dfrac{z-7}{-2}$ is
(a) 4 (b) 5 (c) 6 (d) 7

7. The number of equivalence relations on the set $\{1, 2, 3\}$ containing $(1, 2)$ and $(2, 1)$ is
(a) 3 (b) 1 (c) 2 (d) None of these

8. Let $A = \left\{x \in R : x \geq \dfrac{1}{2}\right\}$ and $B = \left\{x \in R : x \geq \dfrac{3}{4}\right\}$
If $f : A \to B$ is defined as $f(x) = x^2 - x + 1$, then the solution set of the equation $f(x) = f^{-1}(x)$ is
(a) $\{1\}$ (b) $\{2\}$ (c) $\left\{\dfrac{1}{2}\right\}$ (d) None of these

9. The complex number $z = x + iy$ which satisfy the equation $\left|\dfrac{z-5i}{z+5i}\right| = 1$ lie on
(a) the x-axis (b) the straight line $y = 5$
(c) a circle passing through origin
(d) None of these

10. The equation of the curve satisfying the differential equation $y(x + y^3)dx = x(y^3 - x)dy$ and passing through the point $(1, 1)$ is
(a) $y^3 - 2x + 3x^2y = 0$ (b) $y^3 + 2x + 3x^2y = 0$
(c) $y^3 + 2x - 3x^2y = 0$ (d) None of these

11. If $\lim\limits_{x \to a} \dfrac{a^x - x^a}{x^x - a^a} = -1$, then a is equal to
(a) 0 (b) e
(c) 2 (d) None of these

12. Let $f(x) = \sin x$, $g(x) = x^2$ and $h(x) = \log_e x$.
If $G(x) = hogof(x)$, then $G''(x)$ is equal to
(a) $2\,\text{cosec}^2 x\,\cot x$ (b) $-2\,\text{cosec}^2 x\,\cot x$
(c) $2\,\text{cosec}^2 x$ (d) $-2\,\text{cosec}^2 x$

13. For any integer n, the integral
$\int_0^{\pi} e^{\cos^2 x} \cos^3(2n+1)x\,dx$ has the value
(a) 1 (b) π (c) 2π (d) None of these

14. The area defined by $1 \leq |x-2| + |y+1| \leq 2$ is
(a) 2 (b) 4 (c) 6 (d) None of these

15. $C_0, C_1, C_2, \ldots\ldots, C_{15}$ are the binomial coefficients in the expansion of $(1 + x)^{15}$, then
$\dfrac{C_1}{C_0} + \dfrac{2C_2}{C_1} + \dfrac{3C_3}{C_2} + \ldots\ldots + \dfrac{15C_{15}}{C_{14}}$ is
(a) 32 (b) 64
(c) 128 (d) None of these

16. Let the function, f, g, h are defined from R to R such that $f(x) = x^2 - 1$, $g(x) = \sqrt{x^2+1}$ and $h(x) = \begin{cases} 0 \text{ if } x < 0 \\ x \text{ if } x \geq 0 \end{cases}$, then $(ho(fog))(x)$ is
(a) x (b) x^2 (c) 0 (d) None of these

17. AB is a diameter of a circle and C is any point on the circumference of the circle. Then
(a) the perimeter of ΔABC is maximum when it is isosceles
(b) the area of ΔABC is minimum when it is isosceles
(c) the area of ΔABC is maximum when it is isosceles
(d) None of these

18. If $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$ and vectors $(1, a, a^2)$, $(1, b, b^2)$ and $(1, c, c^2)$ are non coplanar, then the product abc equals
(a) 2 (b) -1 (c) 1 (d) 0

19. If $\vec{a} = 4\hat{i} + 6\hat{j}$ and $\vec{b} = 3\hat{j} + 4\hat{k}$ then the vector form of component of $\vec{a}$ along $\vec{b}$ is
(a) $\frac{18}{10\sqrt{3}}(3\hat{j} + 4\hat{k})$ (b) $\frac{18}{25}(3\hat{j} + 4\hat{k})$
(c) $\frac{18}{\sqrt{3}}(3\hat{j} + 4\hat{k})$ (d) $\frac{18}{25}(4\hat{i} + 6\hat{j})$

20. Let $f(x) = \sin x - \tan x$, $x \in (0, \pi/2)$, then tangent drawn to the curve $y = f(x)$ at any point will
(a) lie above the curve (b) lie below the curve
(c) nothing can be said (d) be parallel to a fixed line

21. If $[x]$ denotes the greatest integer $\le x$, then the value of $\lim_{x \to 0} |x|^{[\cos x]}$ is
(a) 0 (b) 1
(c) -1 (d) does not exists

22. The differential equation of all conics whose centre lies at origin is of order
(a) 2 (b) 3 (c) 4 (d) None of these

23. Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors such that $\vec{a} \times \vec{b} = \vec{c}$ and $\vec{c} \times \vec{a} = \vec{b}$, then
(a) $\vec{a} \cdot \vec{b} = |\vec{c}|^2$ (b) $\vec{c} \cdot \vec{a} = |\vec{b}|^2$
(c) $\vec{b} \cdot \vec{c} = |\vec{a}|^2$ (d) $\vec{a} \,||\, (\vec{b} \times \vec{c})$

24. If $\cos^{-1}x > \sin^{-1}x$, then x belongs to the interval
(a) $(-\infty, 0)$ (b) $(-1, 0)$
(c) $\left[0, \frac{1}{\sqrt{2}}\right)$ (d) $\left[-1, \frac{1}{\sqrt{2}}\right)$

25. The number of non-zero diagonal matrices of order 4 satisfying $A^2 = A$ is
(a) 2 (b) 4 (c) 16 (d) 15

26. Find the number of ways in which 52 cards can be divided into 4 sets, three of them having 17 cards each and the fourth one having just one card.
(a) $\frac{\lfloor 52}{(\lfloor 17)^3}$ (b) $\frac{\lfloor 52}{(\lfloor 17)^3 \lfloor 3}$ (c) $\frac{\lfloor 51}{(\lfloor 17)^3}$ (d) $\frac{\lfloor 51}{(\lfloor 17)^3 \lfloor 3}$

27. If R is a relation on a finite set having n elements, then the number of relations on A is
(a) 2^n (b) 2^{n^2} (c) n^2 (d) n^n

28. The area bounded by the curve $y = f(x)$, x-axis and the ordinates $x = l$ and $x = b$ is $(b - 1)\sin(3b + 4)$. Then $f(x)$ is
(a) $(x - 1)\cos(3x + 4)$
(b) $\sin(3x + 4)$
(c) $\sin(3x + 4) + 3(x - 1)\cos(3x + 4)$
(d) None of these

29. If the angles of a triangle are in the ratio 4 : 1 : 1, then the ratio of the longest side and perimeter is
(a) $\sqrt{3} : (2 + \sqrt{3})$ (b) 1 : 6
(c) $1 : (2 + \sqrt{3})$ (d) 2 : 6

30. If $|z - 1| + |z + 3| = 8$, then the range of values of $|z - 4|$ is
(a) (0, 8) (b) [0, 8] (c) [1, 9] (d) [5, 9]

31. If a, b, c are the roots of equation $x^3 - px^2 + qx - r = 0$, then the value of $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$ is
(a) $\frac{p^2 - 2qr}{r^2}$ (b) $\frac{q^2 - 2pr}{r^2}$
(c) $\frac{r^2 - 2pq}{q^2}$ (d) $\frac{r^2 - 2pq}{p^2}$

32. If $\int f(x)dx = F(x)$, then $\int x^3 f(x^2)dx$ is equal to
(a) $\frac{1}{2}\{x^2F(x^2) - \int F(x^2)dx^2\}$
(b) $\frac{1}{2}\{x^2F(x^2) - \int F(x^2)dx\}$
(c) $\frac{1}{2}\left(x^2F(x) - \frac{1}{2}\int F(x^2)dx\right)$
(d) None of these

33. In three dimensional space $x^2 - 5x + 6 = 0$ represents
(a) two points (b) two parallel planes
(c) two parallel lines (d) a pair of non parallel lines

34. If $f(x) = \log_x(\log_x)$, then $\frac{d}{dx}(f(x))$ is
(a) $\frac{1}{e}$ (b) e
(c) $e + 1$ (d) None of these

35. Let $f(x) = \begin{cases} \dfrac{x^3 + x^2 - 16x + 20}{(x-2)^2}, & \text{if } x \neq 2. \\ b, & \text{if } x = 2 \end{cases}$

If $f(x)$ is continuous for all x, then b is equal to
(a) 7 (b) 3 (c) 2 (d) 5

36. The locus of the middle points of the chords of the circle $x^2 + y^2 = a^2$ which subtend a right angle at the centre is
(a) $x^2 + y^2 = \dfrac{a^2}{2}$ (b) $x^2 + y^2 = 2a^2$
(c) $x^2 + y^2 = \dfrac{a^2}{4}$ (d) None of these

37. An integrating factor of the differential equation $y \log y \dfrac{dx}{dy} + x - \log y = 0$ is
(a) $\log(\log y)$ (b) $\log y$
(c) $\dfrac{1}{\log y}$ (d) $\dfrac{1}{\log(\log y)}$

38. A fair die is rolled. Consider the events $A = \{1, 3, 5\}$, $B = \{2, 3\}$ and $C = \{2, 3, 4, 5\}$. Then the conditional probability $P((A \cup B) \mid C)$ is
(a) $\dfrac{1}{4}$ (b) $\dfrac{5}{4}$ (c) $\dfrac{1}{2}$ (d) $\dfrac{3}{4}$

39. India play two matches each with West Indies and Australia. In any match, the probabilities of India getting 0, 1 and 2 points are 0.45, 0.05 and 0.50 respectively. Assuming that the outcomes are independent, the probability of India getting at least 7 points is
(a) 0.0875 (b) $\dfrac{1}{16}$
(c) 0.1125 (d) None of these

40. The region represented by the inequation system $x, y \geq 0, y \leq 6, x + y \leq 3$ is
(a) unbounded in first quadrant
(b) unbounded in first and second quadrant
(c) bounded in first quadrant
(d) None of these

41. If $x^2 + px + 1$ is a factor of $ax^3 + bx + c$, then
(a) $a^2 + c^2 = ab$ (b) $a^2 + c^2 = -ab$
(c) $a^2 - c^2 = ab$ (d) None of these

42. The set of values of x for which $\dfrac{\tan 3x - \tan 2x}{1 + \tan 3x \cdot \tan 2x} = 1$ is
(a) ϕ (b) $\left\{\dfrac{\pi}{4}\right\}$
(c) $\left\{n\pi + \dfrac{\pi}{4}, n = 1, 2, 3, \ldots\ldots\right\}$
(d) $\left\{2n\pi + \dfrac{\pi}{4}, n = 1, 2, 3, \ldots\ldots\right\}$

43. If $y = a \log x + bx^2 + x$ has its extremum values at $x = -1$ and $x = 2$, then
(a) $a = 2, b = -1$ (b) $a = 2, b = \dfrac{1}{2}$
(c) $a = -2, b = \dfrac{1}{2}$ (d) $a = 2, b = -\dfrac{1}{2}$

44. The angle through which the axes must be rotated, without translation, in anti-clockwise sense so that the expression $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ does not contain the mixed product xy, is given by
(a) $\tan^{-1}\left(\dfrac{2h}{a-b}\right)$ (b) $\dfrac{1}{2}\tan^{-1}\left(\dfrac{2h}{b-a}\right)$
(c) $\dfrac{1}{2}\tan^{-1}\left(\dfrac{2h}{a-b}\right)$ (d) $\dfrac{1}{2}\tan^{-1}\left(\dfrac{h}{a-b}\right)$

45. If $\cos A + \cos C = 4\sin^2\dfrac{B}{2}$, then the sides a, b, c of the triangle are in
(a) A.P. (b) G.P.
(c) H.P. (d) None of these

46. The two consecutive terms in the expansion of $(3 + 2x)^{74}$ whose coefficients are equal are
(a) 30^{th} and 31^{st} terms (b) 31^{st} and 32^{nd} terms
(c) 29^{th} and 30^{th} terms (d) None of these

47. If $g(f(x)) = |\sin x|$, $f(g(x)) = (\sin\sqrt{x})^2$, then
(a) $f(x) = \sin x$, $g(x) = |x|$
(b) $f(x) = x^2$, $g(x) = \sin\sqrt{x}$
(c) f and g can not be determined
(d) $f(x) = \sin^2 x$, $g(x) = \sqrt{x}$

48. Let $*$ be a binary operation on the set Q^+ of all +ve rational numbers defined by $a * b = \dfrac{ab}{100}$ for all $a, b \in Q^+$. The inverse of 0.1 under opeation $*$ is
(a) 10^5 (b) 10^6
(c) 10^4 (d) None of these

49. If a, b, c, d and p are distinct real numbers such that $(a^2 + b^2 + c^2)p^2 - 2(ab + bc + cd)p + (b^2 + c^2 + d^2) \leq 0$, then a, b, c, d are in
(a) A.P (b) H.P (c) $ab = cd$ (d) G.P

50. If the position vectors of three points are $\vec{a} - 2\vec{b} + 3\vec{c}$, $2\vec{a} + 3\vec{b} - 4\vec{c}$ and $-7\vec{a} + 10\vec{c}$, then the three points are
(a) collinear (b) non-collinear
(c) coplanar (d) None of these

SOLUTIONS

1. (b) : We have, $A = \begin{bmatrix} 1 & 1 & \cdots 1 \\ 1 & 1 & \cdots 1 \\ \vdots & \vdots & \vdots \\ 1 & 1 & \cdots 1 \end{bmatrix}_{m \times n}$

Applying $R_k \to R_k - R_1$ (where $k = 2, 3, ..., m$), we get

$$A = \begin{bmatrix} 1 & 1 & \cdots 1 \\ 0 & 0 & \cdots 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & \cdots 0 \end{bmatrix}_{m \times n}$$

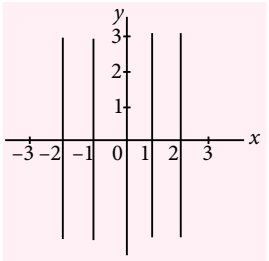
$\therefore$ Matrix A has only one non-zero row.
Hence, rank of matrix $A = 1$

2. (c) : We have,

$$\sqrt{(x-2)^2 + y^2} + \sqrt{(x+2)^2 + y^2} = 4$$

$$\Rightarrow \quad \sqrt{x^2 + 4 - 4x + y^2} + \sqrt{x^2 + 4 + 4x + y^2} = 4$$

By hit and trial method, we get
$x = \pm 2, y = 0$
$x = \pm 1, y = 0$
and $x = 0, y = 0$
Thus, given equation represents a pair of lines (as shown in figure).



3. (b) : Let $z = \lim\limits_{n\to\infty} \dfrac{(n!)^{1/n}}{n} = \lim\limits_{n\to\infty} \left(\dfrac{n!}{n^n}\right)^{1/n}$

$$\Rightarrow \quad z = \lim_{n\to\infty} \left(\frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3}{n} \cdot ..... \frac{n}{n}\right)^{1/n}$$

Taking log on both sides, we get

$$\log z = \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \log\left(\frac{r}{n}\right)$$

$$\Rightarrow \quad \log z = \int_0^1 \log x \, dx$$

$$\Rightarrow \quad \log z = -1 \Rightarrow z = e^{-1}$$

4. (a) : Let $I = \int_0^\infty \dfrac{dx}{(x^2+4)(x^2+9)}$

$$\Rightarrow \quad I = \int_0^\infty \frac{x^2 \, dx}{(x^2+2^2)(x^2+3^2)(x^2+0^2)}$$

$$\Rightarrow \quad I = \frac{\pi}{2(2+3)(3+0)(0+2)} = \frac{\pi}{60}$$

5. (a) : Since A.M. $\geq$ G.M. $\forall$ real x

$$\Rightarrow \quad \frac{4^{\sin^2 x} + 4^{\cos^2 x}}{2} \geq \sqrt{4^{\sin^2 x + \cos^2 x}}$$

$$\Rightarrow \quad 4^{\sin^2 x} + 4^{\cos^2 x} \geq 2\sqrt{4}$$

$$\Rightarrow \quad 4^{\sin^2 x} + 4^{\cos^2 x} \geq 4$$

6. (d)

7. (c) : Equivalence relation of the set {1, 2, 3} containing (1, 2) and (2, 1) are
$A_1 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$
$A_2 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 1), (3, 2), (1, 3)\}$
So, only two equivalence relations exist.

8. (a)

9. (a) : We have $\left|\dfrac{z-5i}{z+5i}\right| = 1 \Rightarrow |z - 5i| = |z + 5i|$

$$\Rightarrow \quad \sqrt{(x)^2 + (y-5)^2} = \sqrt{x^2 + (y+5)^2}$$

$$\Rightarrow \quad x^2 + y^2 + 25 - 10y = x^2 + y^2 + 25 + 10y$$

$$\Rightarrow \quad y = 0 \Rightarrow \quad z \text{ lies on } x\text{-axis.}$$

10. (c) : We have, $y(x + y^3)dx = x(y^3 - x)dy$

$$\Rightarrow \quad y^3(y\,dx - x\,dy) + x(y\,dx + x\,dy) = 0$$

$$\Rightarrow \quad x^2y^3\left(\frac{y\,dx - x\,dy}{x^2}\right) + x\,d(xy) = 0$$

$$\Rightarrow \quad -\frac{y}{x} d\left(\frac{y}{x}\right) + \frac{d(xy)}{x^2y^2} = 0$$

On integrating, we get

$$\frac{-\left(\frac{y}{x}\right)^2}{2} - \frac{1}{xy} + c' = 0 \Rightarrow \frac{y^2}{2x^2} + \frac{1}{xy} + c = 0$$

$$\Rightarrow \quad y^3 + 2x + 2cx^2y = 0$$

Since it passes through the point (1, 1), so

$$1 + 2 + 2c = 0 \Rightarrow \quad c = \frac{-3}{2}$$

$\therefore$ The curve is $y^3 + 2x - 3x^2y = 0$

11. (d) : We have, $\lim\limits_{x\to a} \dfrac{a^x - x^a}{x^x - a^a} = -1 \quad \left(\dfrac{0}{0} \text{ form}\right)$

$$\Rightarrow \quad \lim_{x\to a} \frac{a^x \log a - ax^{a-1}}{x^x(1 + \log x)} = -1$$

$$\Rightarrow \quad \frac{a^a \log a - aa^{a-1}}{a^a(1 + \log a)} = -1$$

$$\Rightarrow \quad \log a - 1 = -\log a - 1 \Rightarrow \log a = 0 \Rightarrow a = 1$$

12. (d) : $f(x) = \sin x$, $g(x) = x^2$ and $h(x) = \log_e x$
$G(x) = hogof(x) = hog(\sin x) = h(\sin x)^2$

$$\Rightarrow \quad G(x) = \log_e(\sin x)^2 = 2\log_e \sin x$$

Differentiating both sides, we get

$$G'(x) = 2 \times \frac{1}{\sin x} \times \cos x = 2\cot x$$

Again differentiating, we get
$G''(x) = -2\text{cosec}^2 x$

13. (d) : Let $I = \int_0^\pi e^{\cos^2 x} \cos^3(2n+1)x \, dx$ ...(i)

$$\Rightarrow \quad I = \int_0^\pi e^{\cos^2(\pi - x)} \cos^3(2n+1)(\pi - x)\, dx$$

$$\Rightarrow \quad I = \int_0^{\pi} e^{\cos^2 x} \cos^3((2n+1)\pi - (2n+1)x)\,dx$$

$$= -\int_0^{\pi} e^{\cos^2 x} \cos^3(2n+1)x\,dx \qquad ...(ii)$$

$(\because \cos(n\pi - \theta) = -\cos\theta$ if $n \in$ odd integer)

Adding (i) and (ii), we get

$2I = 0 \Rightarrow I = 0$

14. (c) 15. (d) :

$$\frac{C_1}{C_0} + \frac{2C_2}{C_1} + \frac{3C_3}{C_2} + ... + \frac{15C_{15}}{C_{14}}$$

$$= 15 + (2 \times 7) + 3\left(\frac{13}{3}\right) + ... + 15\left(\frac{1}{15}\right)$$

$$= 15 + 14 + 13 + ... + 1 = \frac{15(15+1)}{2} = 120$$

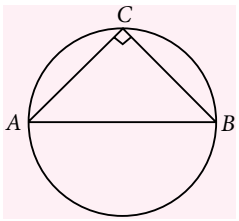
16. (b) : $(ho(fog))x = hof(g(x)) = hof(\sqrt{x^2+1})$

$$= h\left[\left(\sqrt{x^2+1}\right)^2 - 1\right] = h(x^2) = x^2$$

17. (c) : AB is the diameter of the circle and C is any point on the circumference.

So, $\angle ACB = 90°$

$\therefore \Delta ACB$ is a right angle triangle and area of right angled triangle is maximum when it is isosceles.



18. (b) : We have, $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + abc \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow (abc+1)\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = 0$$

$\because (1, a, a^2)$ $(1, b, b^2)$ and $(1, c, c^2)$ are non-coplanar.

$\Rightarrow \quad abc + 1 = 0 \Rightarrow abc = -1$

19. (b) : Vector form of component of $\vec{a}$ along $\vec{b}$

$$= \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right)\hat{b}$$

$$= \left[\frac{(4\hat{i}+6\hat{j}).(3\hat{j}+4\hat{k})}{\sqrt{3^2+4^2}}\right]\left(\frac{3\hat{j}+4\hat{k}}{\sqrt{3^2+4^2}}\right)$$

$$= \left(\frac{0+18+0}{5}\right)\left(\frac{(3\hat{j}+4\hat{k})}{5}\right) = \frac{18}{25}(3\hat{j}+4\hat{k})$$

20. (a) : We have, $f(x) = \sin x - \tan x$

Differentiating both sides, we get

$f'(x) = \cos x - \sec^2 x \qquad ...(i)$

$f'(x) \geq 0 \ \forall \ x \in (0, \pi/2)$

Now, again differentiating (i), we get

$f''(x) = -\sin x - 2\sec x\,(\sec x \tan x)$

$= -(\sin x + 2\sec^2 x \tan x)$

$f''(x) < 0 \ \forall \ x \in (0, \pi/2)$

$\Rightarrow$ $f(x)$ is concave downward.

Hence tangent will lie above the curve.

21. (b) : We have, $\lim_{x\to 0} |x|^{[\cos x]} \qquad ...(i)$

When $x \to 0$, then $0 \leq \cos x \leq 1$

$\Rightarrow \quad [\cos x] = 0$ when $x \neq 0$

From (i), we have $\lim_{x\to 0} |x|^0 = \lim_{x\to 0} 1 = 1$

22. (b) : Consider the general equation of all conics is

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad ...(i)$

As centre of the conic is (0, 0)

$\Rightarrow \quad g = 0, f = 0$

$\therefore$ (i) becomes $ax^2 + 2hxy + by^2 = -c$

or $\left(\frac{a}{-c}\right)x^2 + \left(\frac{2h}{-c}\right)xy + \left(\frac{b}{-c}\right)y^2 = 1$

or $Ax^2 + 2Hxy + By^2 = 1$ is the general equation of all conics whose centre lie at origin.

Since, it has three arbitrary constants.

Hence, order of the differential equation is 3.

23. (d) : We have, $\vec{a} \times \vec{b} = \vec{c} \Rightarrow \vec{a} \perp \vec{c}$ and $\vec{b} \perp \vec{c} \qquad ...(i)$

Also, $\vec{c} \times \vec{a} = \vec{b} \Rightarrow \vec{c} \perp \vec{b}$ and $\vec{a} \perp \vec{b} \qquad ...(ii)$

From (i) and (ii), we get $\vec{a} \perp \vec{c}$ and $\vec{a} \perp \vec{b}$

$\Rightarrow \quad \vec{a} \,||\, (\vec{b} \times \vec{c})$

24. (d) : We know that,

$0 < \cos^{-1}x < \pi$ and $\frac{-\pi}{2} < \sin^{-1}x < \frac{\pi}{2}$

Also, $\cos^{-1}x$ is decreasing function and $\sin^{-1}x$ is increasing function.

And $\sin^{-1}x = \cos^{-1}x = \frac{\pi}{4}$

$\therefore \quad \cos^{-1}x > \sin^{-1}x$ when

$$\cos^{-1}x \in \left[\frac{\pi}{4}, \pi\right) \Rightarrow x \in \left[-1, \frac{1}{\sqrt{2}}\right)$$

25. (d)

26. (b) : We know that if $3m$ things are divided into three equal groups, then the number of divisions

$$= \frac{\lfloor 3m}{\lfloor m \ \lfloor m \lfloor m \lfloor 3}$$

Using this result, 52 cards are divided into 3 equal sets (each containing 17 cards) and 1 set (containing only one card) represented as $\frac{\lfloor 52}{\lfloor 17 \ \lfloor 17 \lfloor 17 \lfloor 3 \lfloor 1} = \frac{\lfloor 52}{(\lfloor 17)^3 \lfloor 3}$

27. (b) : Number of elements in $A = n$
Number of elements in $A \times A = n \times n = n^2$
$\therefore$ Number of relations R in A are subsets of $A \times A$.

$\Rightarrow$ Number of relations $= 2^{n^2}$

28. (c) : The area bounded by the curve $y = f(x)$ with ordinates $x = l$ and $x = b$ is

$$\int_l^b f(x)\,dx = (b-1)\sin(3b+4)$$

Differentiating w.r.t. b, we get
$f(b) = \sin(3b + 4) + (b - 1)\cos(3b + 4) \times 3$
$\Rightarrow \quad f(x) = \sin(3x + 4) + 3(x - 1)\cos(3x + 4)$

29. (a) : Given that angles of the triangle are in the ratio 4 : 1 : 1
$\therefore$ Angles are 120°, 30° and 30°.
By sine rule, we have

$$\frac{\sin 120^\circ}{a} = \frac{\sin 30^\circ}{b} = \frac{\sin 30^\circ}{c} = \gamma \text{ (say)}$$

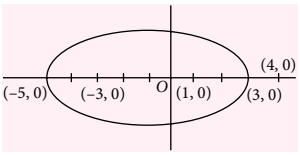
$$\Rightarrow \quad a = \frac{\sin 120^\circ}{\gamma} = \frac{\sqrt{3}}{2\gamma}, \quad b = \frac{\sin 30^\circ}{\gamma} = \frac{1}{2\gamma},$$

$$c = \frac{\sin 30^\circ}{\gamma} = \frac{1}{2\gamma}$$

$$\therefore \text{ Perimeter } = \frac{\sqrt{3}}{2\gamma} + \frac{1}{2\gamma} + \frac{1}{2\gamma} = \left(\frac{\sqrt{3}+2}{2\gamma}\right)$$

$$\text{Now, required ratio } = \frac{\sqrt{3}}{2\gamma} : \frac{(\sqrt{3}+2)}{2\gamma} = \sqrt{3} : (\sqrt{3}+2)$$

30. (c) : Given that $|z - 1| + |z + 3| = 8$
$\Rightarrow$ z lies on the ellipse whose foci are (1, 0) and (–3, 0)
$\therefore$ Vertices of the ellipse are (3, 0) and (–5, 0)
Hence, the maximum and minimum values of $|z - 4|$ is 9 and 1 respectively.
$\therefore$ Range of $|z - 4|$ is [1, 9].



31. (b)

32. (a) : Let $I = \int x^3 f(x^2)\,dx$

Put $x^2 = t \ \Rightarrow \ 2x\,dx = dt$

$$\Rightarrow \quad I = \int t\,f(t)\frac{dt}{2} = \frac{1}{2}\int t\,f(t)\,dt$$

$$\Rightarrow \quad I = \frac{1}{2}\left[t\,F(t) - \int F(t)\,dt\right]$$

$$= \frac{1}{2}\left[x^2F(x)^2 - \int F(x^2)\,dx^2\right]$$

33. (b) : We have, $x^2 - 5x + 6 = 0$
$\Rightarrow \quad (x - 3)(x - 2) = 0 \ \Rightarrow x - 3 = 0$ or $x - 2 = 0$
In three dimensional space, these two equations represent two parallel planes.

34. *(Question is not correct)*

35. (a) : $f(x) = \begin{cases} \dfrac{x^3 + x^2 - 16x + 20}{(x-2)^2}, & \text{if } x \neq 2 \\ b, & \text{if } x = 2 \end{cases}$

$$\therefore \quad \lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^3 + x^2 - 16x + 20}{(x-2)^2}$$

$$= \lim_{x\to 2} \frac{(x-2)(x^2+3x-10)}{(x-2)^2}$$

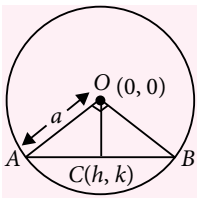
$$= \lim_{x\to 2} \frac{(x-2)(x+5)(x-2)}{(x-2)^2}$$

$$= \lim_{x\to 2} (x+5) = 2 + 5 = 7$$

$\because$ $f(x)$ is continuous for all x.

$$\therefore \quad f(2) = \lim_{x\to 2} f(x) \Rightarrow b = 7$$

36. (a) : Circle $x^2 + y^2 = a^2$ having centre at origin and radius a. Let AB be the chord of the circle which subtend a right angle at the centre and $C(h, k)$ be the mid-point of the chord.



Now, in ΔACO, $\sin 45^\circ = \frac{OC}{OA}$

$$\Rightarrow \quad \frac{1}{\sqrt{2}} = \frac{OC}{a} \Rightarrow \quad OC = \frac{a}{\sqrt{2}}$$

$$\Rightarrow \quad \sqrt{(0-h)^2 + (0-k)^2} = \frac{a}{\sqrt{2}}$$

$$\Rightarrow \quad h^2 + k^2 = \frac{a^2}{2}$$

Hence, locus of $C(h, k)$ is $x^2 + y^2 = \frac{a^2}{2}$

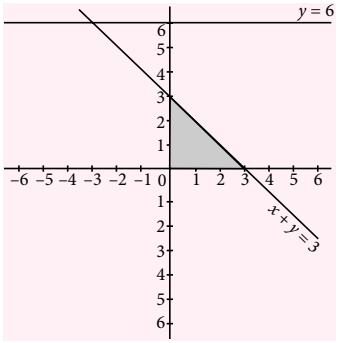
37. (b)

38. (d) : $A = \{1, 3, 5\}$, $B = \{2, 3\}$, $C = \{2, 3, 4, 5\}$
$\Rightarrow \quad A \cup B = \{1, 2, 3, 5\}$.
and $(A \cup B) \cap C = \{2, 3, 5\}$

$$\text{Now, } P[(A \cup B)|C] = \frac{n\left[(A \cup B) \cap C\right]}{n(C)} = \frac{3}{4}$$

39. (a) : India will get atleast 7 points.
$\therefore$ India get 2 points in all 4 matches or 2 points in 3 matches and 1 point in one match.
Thus, probability of getting at least 7 points
$= (0.5)^4 + {}^4C_3(0.5)^3 \times 0.05 = 0.0875$

40. (c) :



Hence, area of the given inequality is bounded in first quadrant.

41. (c) : $\because$ $x^2 + px + 1$ is the factor of $ax^3 + bx + c = 0$ then let α, β be the roots of $x^2 + px + 1 = 0$.

$\Rightarrow \quad \alpha + \beta = -p$ ...(i)

$\Rightarrow \quad \alpha\beta = 1$ ...(ii)

Now, α, β, γ be the roots of $ax^3 + bx + c = 0$.

$\Rightarrow \quad \alpha + \beta + \gamma = 0$ ...(iii)

$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{b}{a}$ ...(iv)

$\alpha\beta\gamma = \frac{-c}{a}$ ...(v)

From (i) and (iii), we get $\gamma = p$

From (ii) and (v), we get $\gamma = -\frac{c}{a}$

Now, (iv) becomes

$$1 + \beta p + p\alpha = \frac{b}{a} \Rightarrow p(\alpha + \beta) = \frac{b}{a} - 1$$

$$\Rightarrow p(-p) = \frac{b-a}{a} \Rightarrow -p^2 = \frac{b-a}{a}$$

$$\Rightarrow -\gamma^2 = \frac{b-a}{a} \Rightarrow -\left(\frac{-c}{a}\right)^2 = \frac{b-a}{a}$$

$$\Rightarrow \frac{-c^2}{a^2} = \frac{b-a}{a} \Rightarrow a^2 - c^2 = ab$$

42. (a) : We have, $\frac{\tan 3x - \tan 2x}{1 + \tan 3x \cdot \tan 2x} = 1$

$\Rightarrow \quad \tan(3x - 2x) = 1 \Rightarrow \tan x = 1 = \tan\left(\frac{\pi}{4}\right)$

$\Rightarrow \quad x = n\pi + \frac{\pi}{4}, n \in Z \cdot$ But if $x = n\pi + \frac{\pi}{4}$, then

$$\tan 2x = \tan 2\left(n\pi + \frac{\pi}{4}\right) = \tan\left(2n\pi + \frac{\pi}{2}\right) = \tan\left(\frac{\pi}{2}\right) = \infty$$

Thus, there is no value exist for which given equation holds. Hence ,solution set = ϕ.

43. (d) : Given, $y = a \log x + bx^2 + x$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{a}{x} + 2bx + 1$$

$\because$ y has extremum values at $x = -1$ and $x = 2$.

Hence, $\left[\frac{dy}{dx}\right]_{-1} = 0 \Rightarrow -a - 2b + 1 = 0$ ...(i)

and $\left[\frac{dy}{dx}\right]_{2} = 0 \Rightarrow a + 8b + 2 = 0$ ...(ii)

Solving (i) and (ii), we get $a = 2$ and $b = \frac{-1}{2}$

44. (c) **45. (a)**

46. (a) : Let r^{th} and $(r+1)^{th}$ terms of $(3 + 2x)^{74}$ has equal coefficient.

$\Rightarrow$ Coefficient of T_r = Coefficient of T_{r+1}

$\Rightarrow \quad {}^{74}C_{r-1}\,(3)^{75-r}\,2^{r-1} = {}^{74}C_r\,(3)^{74-r}\,2^r$

$$\Rightarrow \frac{{}^{74}C_r}{{}^{74}C_{r-1}} = \frac{3}{2} \Rightarrow \frac{74 - r + 1}{r} = \frac{3}{2}$$

$\Rightarrow \quad 5r = 150 \Rightarrow r = 30$

Hence, required terms are 30^{th} and 31^{st}.

47. (d) : From option (d), we get

$$g(f(x)) = g(\sin^2 x) = \sqrt{\sin^2 x} = |\sin x|$$

and $f(g(x)) = f(\sqrt{x}) = \sin^2\sqrt{x} = (\sin\sqrt{x})^2$

48. (a) : We have, $a * b = \frac{ab}{100}$

Let e be the identity element, then

$e * a = a * e = a$

$$\Rightarrow \quad a = \frac{ea}{100} \Rightarrow e = 100$$

Now, let inverse of 0.1 be x, then

$$0.1 * x = e \Rightarrow \frac{0.1\,x}{100} = 100 \Rightarrow x = 100000 = 10^5$$

49. (d) : We have

$(a^2 + b^2 + c^2)p^2 - 2(ab + bc + cd)\,p + (b^2 + c^2 + d^2) \le 0$

$\Rightarrow \quad (a^2p^2 - 2abp + b^2) + (b^2p^2 - 2bcp + c^2) +$

$(c^2p^2 - 2cdp + d^2) \le 0$

$\Rightarrow \quad (ap - b)^2 + (bp - c)^2 + (cp - d)^2 \le 0$

This is possible only if $ap - b = bp - c = cp - d = 0$

$$\Rightarrow \frac{b}{a} = \frac{c}{b} = \frac{d}{c} = p \Rightarrow a, b, c, d \text{ are in G.P.}$$

50. (a,c): Let λ_1, λ_2 and λ_3 be the three points such that

$(\vec{a} - 2\vec{b} + 3\vec{c})\,\lambda_1 + (2\vec{a} + 3\vec{b} - 4\vec{c})\lambda_2 +$

$(-7\vec{a} + 10\vec{c})\,\lambda_3 = 0$

$\Rightarrow \quad (\lambda_1 + 2\lambda_2 - 7\lambda_3)\vec{a} + (-2\lambda_1 + 3\lambda_2)\vec{b} +$

$(3\lambda_1 - 4\lambda_2 + 10\lambda_3)\vec{c} = 0$

Equating the coefficients, we get

$\lambda_1 + 2\lambda_2 - 7\lambda_3 = 0$ ...(i)

$-2\lambda_1 + 3\lambda_2 = 0$ ...(ii)

$3\lambda_1 - 4\lambda_2 + 10\lambda_3 = 0$ ...(iii)

On solving (i), (ii) and (iii) we get

$\lambda_1 + \lambda_2 + \lambda_3 = 0$

Hence, given vectors are collinear and coplanar.

CONCEPT BOOSTERS

Class XI

COMPLEX NUMBERS

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This column is aimed at Class XI students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

INTRODUCTION

For the complex number $z = a + ib$, a is called the real part, denoted by $\text{Re}(z)$ and b is called the imaginary part denoted by $\text{Im}(z)$. A complex number z is said to be real if $\text{Im}(z) = 0$ and is said to be purely imaginary if $\text{Re}(z) = 0$. Thus the complex number $0 = 0 + 0i$, is both real and purely imaginary. $a + ib$ is called an imaginary number if $b \neq 0$.

Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ are equal if and only if $a = c$ and $b = d$. However there is no order relation in the set of complex numbers *i.e.*, the expressions of the form $a + ib < c + id$ and $a + ib > c + id$ are meaningless unless $b = d = 0$.

ALGEBRA OF COMPLEX NUMBERS

Let $a, b, c, d, \lambda \in R$, then

(i) Addition : $(a + ib) + (c + id) = (a + c) + i(b + d)$

(ii) Subtraction : $(a + ib) - (c + id) = (a - c) + i(b - d)$

(iii) Multiplication :

$(a + ib)(c + id) = (ac - bd) + i(ad + bc)$

(iv) Division :

For $c + id \neq 0$, $\dfrac{a+ib}{c+id} = \dfrac{ac+bd}{c^2+d^2} + i\dfrac{bc-ad}{c^2+d^2}$

(v) Multiplication by a real number :

$\lambda(a + ib) = \lambda a + i(\lambda b)$

THE CONJUGATE OF A COMPLEX NUMBER

Let $z = a + ib$ be a complex number. The conjugate of z, denoted as $\bar{z}$, is the complex number $a - ib$, *i.e.*, $\bar{z} = a - ib$.

Thus $a = \text{Re}(z) = \dfrac{z+\bar{z}}{2}$ and $b = \text{Im}(z) = \dfrac{z-\bar{z}}{2i}$

Let z, z_1 and z_2 be complex numbers. The following results can easily be proved.

(i) $z = \bar{z} \Leftrightarrow z$ is a real number.

(ii) $z = -\bar{z} \Leftrightarrow z$ is a purely imaginary number.

(iii) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ **(iv)** $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(v) $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$ **(vi)** $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}, z_2 \neq 0$

(vii) $\overline{(\bar{z})} = z$ **(viii)** $\overline{(z^n)} = (\bar{z})^n, n \in I$

(ix) $\overline{f(z)} = f(\bar{z})$, f being a polynomial with real coefficients.

(x) Let $z = x + iy$, then $z\bar{z} = x^2 + y^2$. Thus $z\bar{z}$ is a non-negative real number for any complex number.

NOTE : The imaginary roots of a polynomial equation over reals, occur in conjugate pairs and hence any polynomial equation of odd degree over reals has at least one real root. Obviously this is not the case for the polynomial equations over C, the set of complex numbers.

THE MODULUS OF A COMPLEX NUMBER

Let $z = a + ib$ be a complex number. The modulus of z, denoted by $|z|$, is defined to be a non-negative real number $\sqrt{a^2 + b^2}$, *i.e.* $|z| = \sqrt{a^2 + b^2}$.

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Alternate definition :

Let z be a complex number then $|z|$ can be defined as $(z\bar{z})^{1/2}$.

Let z, z_1 and z_2 be complex numbers, then

(i) $|z| = |\bar{z}| \geq 0$ and equality holds if and only if $z = 0$.

(ii) $\text{Re}(z) \leq |z|$ and equality holds if and only if $\text{Im}(z) = 0$ and $\text{Re}(z) \geq 0$.

(iii) $\text{Im}(z) \leq |z|$ and equality holds if and only if $\text{Re}(z) = 0$ and $\text{Im}(z) \geq 0$.

(iv) $z\bar{z} = |z|^2$ and hence $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$, $z \neq 0$

(v) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\text{Re}(z_1\bar{z}_2)$

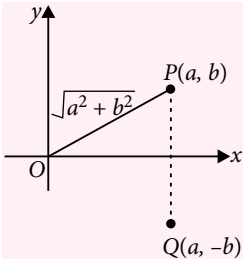
(vi) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\,\text{Re}(z_1\bar{z}_2)$

(vii) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

(viii) $|z_1 z_2| = |z_1| \cdot |z_2|$

(ix) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $z_2 \neq 0$

GEOMETRICAL REPRESENTATION OF COMPLEX NUMBERS



A complex number $a + ib$ corresponds to the ordered pair (a, b), which can be represented geometrically as the unique point (a, b) in the xy plane and vice-versa.

The xy plane having a complex number assigned to each of its points is called the complex plane or the Argand plane.

Let $z = a + ib$. Then z is represented as the point $P(a, b)$ on the argand plane. Obviously $OP = \sqrt{a^2 + b^2} = |z|$. Hence $|z|$ is the distance of z from origin. Further if $Q(a, -b)$ is the mirror image of $P(a, b)$ in the x-axis (also called as real axis), then Q is the geometrical representation of $\bar{z}$. Also note that a complex number is real if and only if it lies on the real axis and a complex number is purely imaginary if and only if it lies on the imaginary axis. For any complex number z_1 and z_2, it is easy to see that $|z_1 - z_2|$ is the distance of z_1 from z_2, when z_1 and z_2 are represented as points on the argand plane.

NOTE : The locus of z satisfying $|z| = r$, $r > 0$ is the circle having centre at origin and radius r. Similarly, locus of z satisfying $|z - z_1| = r$, $r > 0$ is the circle having centre at z_1 and radius r.

The Geometrical representation of $z_1 + z_2$ and $z_1 - z_2$: Let P, Q represent two complex numbers $x_1 + iy_1$ and $x_2 + iy_2$ respectively in the Argand plane. Join the origin O with the points P and Q and complete the parallelogram $OPRQ$. It is clear that the coordinates of the point R are $(x_1 + x_2, y_1 + y_2)$ and hence the point R represents the complex number $(x_1 + x_2) + i(y_1 + y_2) = z_1 + z_2$.

In triangle OPR, OR = distance of R from O = $|z_1 + z_2|$. Similarly $OP = |z_1|$ and $PR = OQ = |z_2|$. Using the triangle inequality in triangle OPR, we get $OR < OP + PR \Rightarrow |z_1 + z_2| < |z_1| + |z_2|$. However if O, P and Q are collinear (P and Q being on the same side of O), then $OR = OP + PR$, *i.e.* $|z_1 + z_2| = |z_1| + |z_2|$.

Hence for any complex numbers z_1, z_2; $|z_1 + z_2| \leq |z_1| + |z_2|$. This is called the triangle inequality.

$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

Its geometrical interpretation can now be explained as $OR^2 + PQ^2 = OP^2 + PR^2 + RQ^2 + QO^2$, in a parallelogram the sum of the squares of the diagonals is equal to the sum of the squares of the four sides.

ARGUMENT OF A COMPLEX NUMBER

Let z be a non-zero complex number, which corresponds to the point P on the Argand plane. If $|z| = r$, then θ satisfies $x = r\cos\theta$, $y = r\sin\theta$, where $z = x + iy$, and θ is called the argument of the given complex number.

The angle of the line segment OP with the positive direction of x-axis is θ. The angle can also be taken as $2n\pi + \theta$, $n \in I$. This angle is called argument or amplitude of the complex number z denoted by $\arg(z)$ or $\text{amp}(z)$. Value of θ having minimum modulus value is called the principal argument of z.

Thus for any non-zero $z \in C$, principal $\arg(z) \in (-\pi, \pi]$. Obviously a non-zero complex number z is real if and only if principal argument of z is 0 or π and it is purely imaginary if and only if principal argument of z is $\pm\frac{\pi}{2}$.

Some Properties of Argument

Let z be a non-zero complex number and $n \in I$

(i) $\arg(-z) = \arg(z) + (2n + 1)\pi$

(ii) $\arg(z + \bar{z}) = n\pi$, z not being purely imaginary.

(iii) $\arg(z - \bar{z}) = (4n \pm 1)\frac{\pi}{2}$, z not being purely real.

(iv) $\arg \bar{z} = \arg\left(\frac{z\bar{z}}{z}\right) = \arg\left(\frac{1}{z}\right) + 2n\pi$

(v) $\arg z + \arg \bar{z} = 2n\pi$

Method of finding principal argument of a non-zero complex number

Let $z = x + iy$ be a given non-zero complex number, whose principal argument is to be found.

Let $\alpha = \tan^{-1}\left|\frac{y}{x}\right|$. Now α, $\pi - \alpha$, $-\pi + \alpha$, $-\alpha$ becomes the principal argument of z according as the point $P(z)$ lies in first, second, third, fourth quadrant respectively.

POLAR FORM OF A COMPLEX NUMBER

Let z be a given non-zero complex number, which corresponds to the point P in the Argand plane. Let $z = x + iy$. Then $OM = x$, $PM = y$. Let $r = |z|$, then $OP = r$. Let $\theta = \arg(z)$, then $\angle POM = \theta$. Now $x = r\cos\theta$ and $y = r\sin\theta$.

Hence $z = x + iy = r(\cos\theta + i\sin\theta)$. This form is called polar form of z, sometimes written in short as $r\text{cis}\theta$. Note that (r, θ) are polar coordinates of z.

NOTE : It is a known fact that

$$\cos\theta = 1 - \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4} - \frac{\theta^6}{\lfloor 6} + \ldots\ldots$$

$$\text{and } \sin\theta = \theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} - \frac{\theta^7}{\lfloor 7} + \ldots\ldots$$

$$\text{Hence } \cos\theta + i\sin\theta = 1 + \frac{i\theta}{\lfloor 1} + \frac{(i\theta)^2}{\lfloor 2} + \frac{(i\theta)^3}{\lfloor 3} + \ldots\ldots$$

$$\text{Also we know that } e^x = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots\ldots$$

Hence, $e^{i\theta} = \cos\theta + i\sin\theta$

Some Consequences of Euler Formula

- Let z be a non-zero complex number and let $re^{i\theta}$ be its polar form. As $\arg(z) = \theta$, so $\arg(z) = 2n\pi + \theta$, $n \in I$. Hence $z = re^{i(\theta + 2n\pi)}$.

$$\text{Now } \log z = \log re^{i(\theta+2n\pi)} = \log r + i(\theta + 2n\pi), n \in I.$$

Thus logarithm of an imaginary number is not unique.

- $\cos\theta + i\sin\theta = e^{i\theta} \Rightarrow \cos(-\theta) + i\sin(-\theta) = e^{-i\theta}$

$$\text{Thus, } \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \text{ and } \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

DE MOIVRE'S THEOREM

We define $e^{i\theta} = \cos\theta + i\sin\theta$.

Let n be any integer, then

$$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos n\theta + i\sin n\theta$$

Thus $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, $n \in I$

This result is called De Moivre's Theorem for an integer. However if $n = \frac{p}{q}, p, q \in I, p \neq 0$ and $q > 1$, then

$$(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{p/q} = \cos\frac{p\theta}{q} + i\sin\frac{p\theta}{q},$$

which is one of the values of $(\cos\theta + i\sin\theta)^{p/q}$.

To make this point more clear, consider $(-1)^{1/2}$ *i.e.*, all the complex numbers whose square is -1, which are i and $-i$, for $-1 = \cos\pi + i\sin\pi$. Hence

$$(-1)^{1/2} = (\cos\pi + i\sin\pi)^{1/2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i,$$

which is one of the value of $(-1)^{1/2}$.

For the other value, we write $-1 = \cos 3\pi + i\sin 3\pi$

$$\Rightarrow \quad (-1)^{1/2} = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = -i$$

NOTE : De Moivre's Theorem is true for irrational values of n also.

Applications of De Moivre's Theorem to Trigonometry

(i) Let n be a positive integer. Then $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$. Also expanding $(\cos\theta + i\sin\theta)^n$ by using Binomial Theorem and comparing the real and imaginary parts, we get

$$\cos n\theta = \cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta - \ldots.$$

$$\sin n\theta = {}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + \ldots.$$

$$\text{Hence, } \tan n\theta = \frac{{}^nC_1\tan\theta - {}^nC_3\tan^3\theta + \ldots\ldots}{1 - {}^nC_2\tan^2\theta + {}^nC_4\tan^4\theta - \ldots\ldots}$$

(ii) Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then

$$z_1z_2 = r_1e^{i\theta_1} \cdot r_2e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)}$$
$$= r_1r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$$

Hence $|z_1z_2| = r_1r_2 = |z_1|\,|z_2|$

and $\arg(z_1z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$

NOTE : (i) Infact, $\arg(z_1z_2) = \arg z_1 + \arg z_2 + 2n\pi$, for some $n \in I$

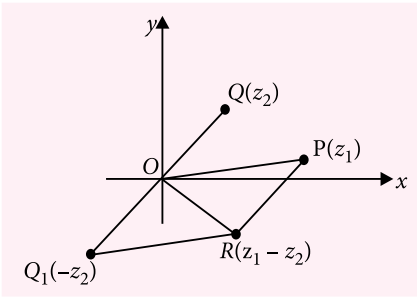
$$\text{Similarly, } \arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 + 2n\pi, \text{ for some } n \in I$$

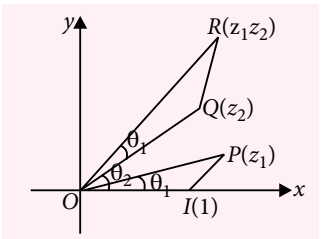
and $\arg(z^n) = n\arg(z) + 2k\pi$, $k \in I$

(ii) Let $z = re^{i\alpha}$. Then $ze^{i\theta} = re^{i(\alpha+\theta)}$. Thus the complex number $ze^{i\theta}$ (in the Argand plane) can be obtained from z by rotating z about origin by an angle θ.

GEOMETRICAL REPRESENTATION OF z_1z_2 AND z_1/z_2

Let P and Q represent two complex numbers z_1 and z_2 respectively in the Argand plane. Join the origin O with the points P and Q. Now we are in search of a point $R(z_1z_2)$ in the Argand plane. For that let I be the point representing the number 1. Now consider a triangle OQR, which is similar to the triangle OIP.





Now argument of the complex number corresponding to R is $\theta_1 + \theta_2 = \arg(z_1z_2)$

Further

$$\frac{OR}{OP} = \frac{OQ}{OI} \Rightarrow OR = \frac{|z_1|\cdot|z_2|}{1}$$

Thus the complex number corresponding to R is

$|z_1|.|z_2|.\ (\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$

$= |z_1|(\cos\theta_1 + i\sin\theta_1)\ |z_2|(\cos\theta_2 + i\sin\theta_2) = z_1z_2$

For the representation of $\frac{z_1}{z_2}$ in the Argand plane once again, let P and Q represent the complex numbers z_1 and z_2 in the Argand plane (as shown in the figure). Now construct a triangle OPR similar to triangle OQI. Then argument of the complex number corresponding to R is

$$\theta_1 - \theta_2 = \arg\left(\frac{z_1}{z_2}\right)$$

Further $\frac{OR}{OP} = \frac{OI}{OQ} \Rightarrow OR = \frac{OP}{OQ} = \frac{|z_1|}{|z_2|} = \left|\frac{z_1}{z_2}\right|$

Hence the complex number corresponding to R is

$$\frac{|z_1|}{|z_2|}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$$

$$= \frac{|z_1|(\cos\theta_1 + i\sin\theta_1)}{|z_2|(\cos\theta_2 + i\sin\theta_2)} = \frac{z_1}{z_2}$$

CONCEPT OF ROTATION

Let z_1, z_2 and z_3 be the vertices of a triangle ABC described in the anticlockwise sense.

$AB = |z_2 - z_1|$, $AC = |z_3 - z_1|$ and

$BC = |z_3 - z_2| = |(z_3 - z_1) - (z_2 - z_1)|$

Now $\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \arg(z_3 - z_1) - \arg(z_2 - z_1) = \alpha$

Also $\left|\frac{z_3 - z_1}{z_2 - z_1}\right| = \frac{AC}{AB}$

Therefore the polar form of $\frac{z_3 - z_1}{z_2 - z_1}$ is $\frac{AC}{AB}(\cos\alpha + i\sin\alpha)$

if $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$, then the triangle ABC is equilateral.

CONDITION FOR THE FOUR POINTS TO BE CONCYCLIC

Let z_1, z_2, z_3 and z_4 be four concyclic points, represented by A, B, C and D in the Argand plane.

As the points A, B, C and D are concyclic,

$$\angle ADB = \angle ACB \Rightarrow \arg\left(\frac{z_2 - z_4}{z_1 - z_4}\right) = \arg\left(\frac{z_2 - z_3}{z_1 - z_3}\right)$$

(using the concept of rotation)

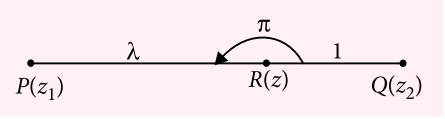
$$\Rightarrow \arg\left(\frac{z_2 - z_4}{z_1 - z_4} \times \frac{z_1 - z_3}{z_2 - z_3}\right) = 0$$

$\Rightarrow$ $\frac{(z_2 - z_4)(z_1 - z_3)}{(z_1 - z_4)(z_2 - z_3)}$ is a positive real number.

NOTE : The converse of the above result is also true, *i.e.*, if a complex number of the form $\frac{(z_2 - z_4)(z_1 - z_3)}{(z_1 - z_4)(z_2 - z_3)}$ is a positive real number, then the points represented by z_1, z_2, z_3 and z_4 on the Argand plane are concyclic.

THE SECTION FORMULA

Let $P(z_1)$ and $Q(z_2)$ be two given points. Let $R(z)$ be the point which divides the join of P and Q internally in $\lambda : 1$ as shown in the figure.



Using the concept of rotation at the point R, we get

$$\frac{z-z_1}{z-z_2}=\lambda e^{i\pi} \quad \Rightarrow \quad z=\frac{z_1+\lambda z_2}{1+\lambda}$$

If R divides PQ externally in $\lambda : 1$, then it can be shown that $z=\frac{z_1-\lambda z_2}{1-\lambda}$.

THE n^{th} ROOTS OF UNITY

Consider the polynomial equation $z^n = 1$.

The roots of this equation are n in number and are called n^{th} roots of unity.

In order to find the roots of $z^n = 1$, we write the polar form of 1, *i.e.*,

$$z^n = \cos 2k\pi + i \sin 2k\pi,\ k \in I$$

$$\Rightarrow \quad z=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}=e^{\frac{2k\pi i}{n}},\ k\in I$$

Now if we give different integral values to k, then we always get roots of $z^n = 1$, but some of them may be repeated. However, if we take $k = 0, 1, 2, 3, ..., n - 1$, then we get (all) distinct roots of $z^n = 1$.

Thus n, n^{th} roots of unity are

$e^{\frac{2k\pi i}{n}}$, $k = 0, 1, 2, ..., n - 1$.

If we take $\alpha=e^{\frac{2\pi i}{n}}$, then the n^{th} roots of unity are $1, \alpha, \alpha^2, ..., \alpha^{n-1}$.

PROPERTIES OF n, n^{th} ROOTS OF UNITY

(i) If n, n^{th} roots of unity are represented on the argand plane, then they form the vertices of a regular n-gon having centre at origin and circumcircle of this n-gon is $|z| = 1$, as modulus value of every root is 1.

(ii) $1 + \alpha + \alpha^2 + ... + \alpha^{n-1} = 0$

(iii) $1 + \alpha^p + (\alpha^p)^2 + ... + (\alpha^p)^{n-1} = 0$, if g.c.d. $(p, n) = 1$.

(iv) $1 \cdot \alpha \cdot \alpha^2 ..... \alpha^{n-1} = (-1)^{n-1}$

(v) $z^n - 1 \equiv (z - 1)(z - \alpha)(z - \alpha)^2 ..... (z - \alpha^{n-1})$

(vi) The imaginary roots form conjugate pair.

EQUATION OF STRAIGHT LINE IN COMPLEX FORM

(i) Equation of line through the points z_1 and z_2 in parametric form is given by

$$z = z_2 + \lambda(z_1 - z_2),\ \lambda \in R$$

(ii) Equation of line through z_1 and z_2 is given by

$$\frac{z-z_1}{z_2-z_1}=\frac{\bar{z}-\bar{z}_1}{\bar{z}_2-\bar{z}_1} \text{ or } \begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix}=0$$

(iii) General equation of a line is given by $\bar{a}z + a\bar{z} + b = 0$, where a is a complex number and b is a real number. Slope of this line is given by $-\frac{\text{Re}(a)}{\text{Im}(a)}$ and its distance from a point z_1 is given by $\left|\frac{\bar{a}z_1+a\bar{z}_1+b}{2|a|}\right|$.

COMPLEX SLOPE OF A LINE

If a line passes through the points z_1 and z_2 then its complex slope is defined as $\frac{z_1-z_2}{\bar{z}_1-\bar{z}_2}$. The complex slope of the line $\bar{a}z + a\bar{z}$ is given by $-\frac{a}{\bar{a}}$.

EQUATION OF A CIRCLE IN COMPLEX FORM

(i) The equation of the circle having centre at z_o and radius r is given by $|z - z_o| = r$

(ii) The general equation of circle is $z\bar{z} + \bar{a}z + a\bar{z} + b = 0$, where a is a complex number and b is a real number. The centre of this circle is '$-a$' and radius is $\sqrt{|a|^2 - b}$.

(iii) The equation of the circle described on the line segment joining z_1 and z_2 as diameter is given by $(z-z_1)(\bar{z}-\bar{z}_2)+(z-z_2)(\bar{z}-\bar{z}_1)=0$.

(iv) $\left|\frac{z-z_1}{z-z_2}\right|=\lambda, \lambda>0, \lambda\neq1$ represents a circle having diameter AB, when A and B divide the join of z_1 and z_2 in $\lambda : 1$ internally and externally respectively.

(v) $\arg\left(\frac{z-z_1}{z-z_2}\right)=\theta$, represents an arc of the circle through z_1 and z_2.

(vi) $|z - z_1|^2 + |z - z_2|^2 = k$, represents a circle provided $k>\frac{1}{2}|z_1-z_2|^2$.

EQUATION OF CONIC SECTION IN COMPLEX FORM

(i) Equation of parabola with focus at z_1 and directrix $a\bar{z} + \bar{a}z + b = 0$ is $|z-z_1|=\left|\frac{a\bar{z}+\bar{a}z+b}{2|a|}\right|$.

(ii) Equation of ellipse with foci at z_1 and z_2 and length of major axis λ, is $|z - z_1| + |z - z_2| = \lambda$.

(iii) Equation of hyperbola with foci at z_1 and z_2 and length of transverse axis λ, is $||z - z_1| - |z - z_2|| = \lambda$.

PROBLEMS

SECTION-I

Single Correct Answer Type

1. If $n \geq 3$ and $1, \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1}$ are n roots of unity, then value of $\sum_{1 \leq i < j \leq n-1} \alpha_i \alpha_j$ is

(a) 0 (b) 1 (c) -1 (d) $(-1)^n$

2. Let $z = \cos\theta + i\sin\theta$. Then, the value of $\sum_{m=1}^{15} \text{Im}(z^{2m-1})$ at $\theta = 2°$ is

(a) $\frac{1}{2°}$ (b) $\frac{1}{3\sin 2°}$ (c) $\frac{1}{2\sin 2°}$ (d) $\frac{1}{4\sin 2°}$

3. If the complex number z satisfying $z + |z| = 2 + 8i$ then value of $|z| =$

(a) 8 (b) 17 (c) 15 (d) 24

4. If $|z + 2 - i| = 5$ then maximum value of $|3z + 9 - 7i| =$

(a) 20 (b) 15 (c) 5 (d) 16

5. If $\lambda \in R$ and non real roots of $2z^2 + 2z + \lambda = 0$ and (0, 0) forms vertices of an equilateral triangle, then λ =

(a) 1 (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) $\frac{2}{3}$

6. If z and w are two complex numbers such that $\bar{z} + i\bar{w} = 0$ and $\arg(zw) = \pi$, then $\arg(z) =$

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{4}$ (d) $\frac{5\pi}{4}$

7. If $A(z_1)$, $B(z_2)$, $C(z_3)$ are vertices of a triangle such that $z_3 = \left(\frac{z_2 - iz_1}{1-i}\right)$ and $|z_1| = 3, |z_2| = 4$ and $|z_2 + iz_1| = |z_1| + |z_2|$, then area of triangle ABC is

(a) $\frac{5}{2}$ (b) 0 (c) $\frac{25}{2}$ (d) $\frac{25}{4}$

8. The radius of the circle given by $\arg\left(\frac{z-5+4i}{z+3-2i}\right) = \frac{\pi}{4}$ is

(a) $5\sqrt{2}$ (b) 5 (c) $\frac{5}{\sqrt{2}}$ (d) $\sqrt{2}$

9. If $f(x) = 2x^3 + 2x^2 - 7x + 72$ then $f\left(\frac{3-5i}{2}\right) =$

(a) 1 (b) 2 (c) 3 (d) 4

10. If $\cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma$ then $\sin^2\alpha + \sin^2\beta + \sin^2\gamma =$

(a) $\frac{1}{2}$ (b) $\frac{3}{2}$ (c) 4 (d) 1

11. If z_1 and z_2 are two complex numbers such that $z_1^2 + z_2^2 \in R$ and $z_1(z_1^2 - 3z_2^2) = 2$, $z_2(3z_1^2 - z_2^2) = 11$, then $z_1^2 + z_2^2 =$

(a) 5 (b) 125 (c) 25 (d) 15

12. If z is a non-real complex number, then the minimum value of $\frac{\text{Im}\, z^5}{\text{Im}^5 z}$ is

(a) -1 (b) -2 (c) -4 (d) -5

13. Let $z_r (1 \leq r \leq 4)$ be complex numbers such that $|z_r| = \sqrt{r+1}$ and $|30z_1 + 20z_2 + 15z_3 + 12z_4|$
$= k|z_1z_2z_3 + z_2z_3z_4 + z_3z_4z_1 + z_4z_1z_2|$
Then the value of k equals

(a) $|z_1z_2z_3|$ (b) $|z_2z_3z_4|$
(c) $|z_4z_1z_2|$ (d) None of these

14. If P and Q are represented by the complex numbers z_1 and z_2 such that $\left|\frac{1}{z_2} + \frac{1}{z_1}\right| = \left|\frac{1}{z_2} - \frac{1}{z_1}\right|$, then the circumcentre of ΔOPQ, (where O is the origin) is

(a) $\frac{z_1 - z_2}{2}$ (b) $\frac{z_1 + z_2}{2}$ (c) $\frac{z_1 + z_2}{3}$ (d) $z_1 + z_2$

15. If α is non real root of $x^7 = 1$, then $1 + 3\alpha + 5\alpha^2 + 7\alpha^3 + \ldots + 13\alpha^6$ is equal to

(a) 0 (b) $\frac{14}{1-\alpha}$ (c) $\frac{14}{\alpha - 1}$ (d) none of these

16. If z_1, z_2 are two complex numbers satisfying the equation $\left|\frac{z_1 + z_2}{z_1 - z_2}\right| = 1$, then $\frac{z_1}{z_2}$ is a number which is

(a) Positive real (b) Negative real
(c) Zero (d) Lying on imaginary axis

17. If z_1, z_2 and z_3 are the vertices of ΔABC, which is not right angled triangle taken in anti-clockwise direction and z_0 is the circumcentre, then $\left(\frac{z_0 - z_1}{z_0 - z_2}\right)\frac{\sin 2A}{\sin 2B} + \left(\frac{z_0 - z_3}{z_0 - z_2}\right)\frac{\sin 2C}{\sin 2B}$ is equal to

(a) 0 (b) 1
(c) -1 (d) 2

18. If 'a' is a complex number such that $|a| = 1$, then the values of 'a' so that equation $az^2 + z + 1 = 0$ has one purely imaginary root is

(a) $a = \cos\theta + i\sin\theta, \theta = \cos^{-1}\left(\frac{\sqrt{5}-1}{2}\right)$

(b) $a = \sin\theta + i\cos\theta, \theta = \cos^{-1}\left(\frac{\sqrt{5}-1}{2}\right)$

(c) $a = \cos\theta + i\sin\theta, \theta = \cos^{-1}\left(\frac{\sqrt{5}-1}{4}\right)$

(d) $a = \sin\theta + i\cos\theta, \theta = \cos^{-1}\left(\frac{\sqrt{5}-1}{4}\right)$

19. Let $A(z_1)$, $B(z_2)$, $C(z_3)$ be the vertices of a triangle oriented in anti-clockwise direction. If $BC : CA : AB = 2 : \sqrt{2} : \sqrt{3}+1$, then the imaginary part of $\left(\frac{z_3 - z_1}{z_2 - z_1}\right)^4$ is

(a) 0 (b) $-7+2\sqrt{6}$

(c) $7-2\sqrt{6}$ (d) cannot be determined

20. If z is a complex number having least absolute value and $|z - 2 + 2i| = 1$, then $z =$

(a) $\left(2-\frac{1}{\sqrt{2}}\right)(1-i)$ (b) $\left(2-\frac{1}{\sqrt{2}}\right)(1+i)$

(c) $\left(2+\frac{1}{\sqrt{2}}\right)(1-i)$ (d) $\left(2+\frac{1}{\sqrt{2}}\right)(1+i)$

21. Sum of common roots of the equation $z^3 + 2z^2 + 2z + 1 = 0$ and $z^{1985} + z^{100} + 1 = 0$ is

(a) -1 (b) 1 (c) 0 (d) 1

22. Let z be a complex number having the argument θ, $0 < \theta < \frac{\pi}{2}$ and satisfying the equation, $|z - 3i| = 3$. Then $\cot\theta - \frac{6}{z} =$

(a) i (b) $-i$ (c) $2i$ (d) $-2i$

23. Let z_1 and z_2 be any two complex numbers then $\left|z_1+\sqrt{z_1^2-z_2^2}\right|+\left|z_1-\sqrt{z_1^2-z_2^2}\right|$ is equal to

(a) $\left|z_1^2-z_2^2\right|+\left|z_1^2+z_2^2\right|$ (b) $|z_1-z_2|+\left|z_1^2+z_2^2\right|$

(c) $|z_1+z_2|+\left|z_1^2+z_2^2\right|$ (d) $|z_1+z_2|+|z_1-z_2|$

24. Both the roots of the equation $z^2 + az + b = 0$ are of unit modulus if

(a) $|a| \le 2$, $|b| = 1$, arg $b = 2$ arg a

(b) $|a| \le 2$, $|b| = 1$, arg $b =$ arg a

(c) $|a| \ge 2$, $|b| = 2$, arg $b = 2$ arg a

(d) $|a| \ge 2$, $|b| = 2$, arg $b =$ arg a

25. If $|z - i| = 1$ and $\arg(z) = \theta$ where $\theta \in \left(0, \frac{\pi}{2}\right)$, then $\cot\theta - \frac{2}{z}$ equals

(a) $2i$ (b) $3i$ (c) i (d) $-i$

SECTION-II

Multiple Correct Answer Type

26. If $z_1 = a + ib$ and $z_2 = c + id$ are complex numbers such that $|z_1| = |z_2| = 1$ and $\text{Re}(z_1\bar{z}_2) = 0$ then the pair of complex numbers $\omega_1 = a + ic$ and $\omega_2 = b + id$ satisfies

(a) $|\omega_1| = 1$ (b) $|\omega_2| = 1$

(c) $\text{Re}(\omega_1\bar{\omega}_2) = 0$ (d) $\omega_1\bar{\omega}_2 = 0$

27. If points A and B are represented by the non-zero complex numbers z_1 and z_2 on the Argand plane such that $|z_1 + z_2| = |z_1 - z_2|$ and O is the origin, then

(a) orthocentre of ΔOAB lies at O

(b) circumcentre of ΔOAB is $\frac{z_1 + z_2}{2}$

(c) $\text{Arg}\left(\frac{z_1}{z_2}\right) = \pm\frac{\pi}{2}$

(d) ΔOAB is isosceles

28. The adjacent vertices of a regular polygon of n sides whose centre is at origin are given by $(1+\sqrt{2}, 1)$, $(1+\sqrt{2}, -1)$. Then the value of n is

(a) 8 (b) 4 (c) 12 (d) 6

29. If α is a variable complex number such that $|\alpha| > 1$ and $z = \alpha + \frac{1}{\alpha}$ lies on a conic then

(a) Eccentricity of the conic is $\frac{2|\alpha|}{1+|\alpha|^2}$

(b) Distance between foci is 4

(c) Length of latus rectum is $\frac{2(|\alpha|^2 - 1)}{|\alpha|^2 + 1}$

(d) Distance between directrix is $\left(|\alpha| + \frac{1}{|\alpha|}\right)^2$

30. If $z = 1 + \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}$ then

(a) $|z| = 2\cos\frac{3\pi}{5}$ (b) $|z| = 2\cos\frac{2\pi}{5}$

(c) $\arg z = \frac{3\pi}{5}$ (d) $\arg z = -\frac{2\pi}{5}$

31. Let $z_1, z_2, z_3, \ldots, z_n$ are the complex numbers such that $|z_1| = |z_2| = \ldots = |z_n| = 1$. If $z = \left(\sum_{k=1}^{n} z_k\right)\left(\sum_{k=1}^{n} \frac{1}{z_k}\right)$ then

(a) z is purely imaginary

(b) z is real

(c) $0 < z \le n^2$

(d) z is a complex number of the form $a + ib$

32. If a, b, c are non – zero complex numbers of equal moduli and satisfy $az^2 + bz + c = 0$ then

(a) $\min|z| = \frac{\sqrt{5}-1}{2}$ (b) min $|z| = 0$

(c) min $|z|$ does not exist (d) $\max|z| = \frac{\sqrt{5}+1}{2}$

33. If $|z - 3| = \min\{|z - 1|, |z - 5|\}$ then $\text{Re}(z) =$

(a) 2 (b) $\frac{5}{2}$ (c) $\frac{7}{2}$ (d) 4

34. The complex slope μ of a line containing the points z_1 and z_2 in the complex plane is defined as

$\frac{z_1 - z_2}{\overline{z_1 - z_2}}$. If μ_1, μ_2 are the complex slopes of two lines L_1 and L_2, then

(a) L_1 and L_2 are perpendicular if $\mu_1 + \mu_2 = 0$
(b) L_1 and L_2 are parallel if $\mu_1 + \mu_2 = 0$
(c) L_1 and L_2 are perpendicular if $\mu_1\mu_2 = -1$
(d) L_1 and L_2 are parallel if $\mu_1 = \mu_2$

SECTION-III

Comprehension Type

Paragraph for Question No. 35 to 37

Let z_1 be a complex number of magnitude unity and z_2 be a complex number given by $z_2 = z_1^2 - z_1$.

35. If $\arg z_1 = \theta$, then $|z_2|$ is equal to

(a) $2\left|\sin\frac{\theta}{2}\right|$ (b) $2\left|\cos\frac{\theta}{2}\right|$

(c) $\sqrt{2}\left|\sin\frac{\theta}{2}\right|$ (d) $\sqrt{2}\left|\cos\frac{\theta}{2}\right|$

36. If $\arg z_1 = \theta$ and $4n\pi < \theta < (4n + 2)\pi$ (n is an integer), then $\arg z_2$ is equal to

(a) $\frac{3\theta}{2}$ (b) $\frac{\pi - 3\theta}{2}$

(c) $\frac{\pi + 3\theta}{2}$ (d) $\frac{\pi + \theta}{2}$

37. If $\arg z_1 = \theta$ and $(4n + 2)\pi < \theta < (4n + 4)\pi$ (n is an integer), then $\arg z_2$ is equal to

(a) $\frac{\pi}{2} + 3\theta$ (b) $\frac{3\pi}{2} + \frac{3\theta}{2}$

(c) $\frac{3\pi}{2} + 3\theta$ (d) $\frac{\pi}{2} + \frac{3\theta}{2}$

Paragraph for Question No. 38 to 40

Let $A_1, A_2, A_3, ..., A_n$ be a regular polygon of 'n' sides whose centre is origin O. Let the complex numbers representing vertices $A_1, A_2, A_3, ..., A_n$ be $z_1, z_2, z_3, ..., z_n$ respectively. Let $OA_1 = OA_2 = ... = OA_n = 1$

38. The value of $|A_1A_2|^2 + |A_1A_3|^2 + ... + |A_1A_n|^2 =$

(a) n (b) $2n$
(c) $2(n - 1)$ (d) $2(n + 1)$

39. The distances A_1A_j ($j = 2, 3, ..., n$) must be equal to

(a) $\sin\frac{j\pi}{n}$ (b) $2\sin\frac{(j-1)\pi}{n}$

(c) $\cos\frac{j\pi}{n}$ (d) $2\sin\frac{(j+1)\pi}{n}$

40. The value of $|A_1A_2|\,|A_1A_3|$ ..... $|A_1A_n|$ must be equal to

(a) 1 (b) n (c) $\sqrt{n}$ (d) n^2

Paragraph for Question No. 41 to 43

Consider a complex number $w = \frac{z - i}{2z + 1}$ where $z = x + iy$ and $x, y \in R$

41. If the complex number w is purely imaginary then locus of z is

(a) a straight line
(b) a circle with centre $\left(-\frac{1}{4}, \frac{1}{2}\right)$ and radius $\frac{\sqrt{5}}{4}$
(c) a circle with centre $\left(\frac{1}{4}, -\frac{1}{2}\right)$ and passing through origin
(d) neither a circle nor a straight line

42. If the complex number w is purely real, then locus of z is

(a) a straight line passing through origin
(b) a straight line with gradient 3 and y intercept (-1)
(c) a straight line with gradient 2 and y intercept 1
(d) a circle

43. If $|w| = 1$, then locus of z is

(a) a point circle
(b) an imaginary circle
(c) a real circle
(d) not a circle

Paragraph for Question No. 44 to 46

If $z = x + iy$ and (x, y) is a point represented by the complex number 'z' in the argand plane. $|z_1 - z_2|$ denotes distance between z_1 and z_2 in the argand plane.

44. The complex number $z = x + iy$ which satisfies the equation $\left|\frac{z - 5i}{z + 5i}\right| = 1$ lies on the

(a) x-axis
(b) y-axis
(c) circle with radius 5 and centre at origin
(d) line $y = 5$

45. If $|z| = 5$ then the points representing the complex number $-i + \frac{15}{z}$ lie on the circle with

(a) centre (0, 1) and radius = 3
(b) centre (0, –1) and radius = 3
(c) centre (1, 0) and radius = 5
(d) centre (–1, 0) and radius = 15

46. If $|z - i| \le 2$ and $z_1 = 3 + 4i$, then the maximum value of $|iz + z_1|$ is

(a) $\sqrt{20} - 2$ (b) 9
(c) $\sqrt{20} + 2$ (d) 8

SECTION-IV

Matrix-Match Type

47. Consider complex number $z = \cos\alpha + i\sin\alpha, 0 < \alpha < \frac{\pi}{6}$. Then the argument of

Column-I		Column-II	
(A)	$1 + z^3 =$	(p)	$2\alpha - \frac{\pi}{2}$
(B)	$1 - z^4 =$	(q)	$\frac{\pi}{2} + \frac{\alpha}{2}$
(C)	$\frac{1+z^3}{1-z^4} =$	(r)	$\frac{3\alpha}{2}$
(D)	$\frac{z^4-1}{z^3+1} =$	(s)	$\frac{\pi}{2} - \frac{\alpha}{2}$

48. The complex numbers $z_1, z_2, ..., z_n$ represent the vertices of a regular polygon of n sides, inscribed in a circle of unit radius and $z_3 + z_n = Az_1 + \bar{A}z_2$, $[x]$ be the greatest integer $\le x$. Then

When n equals to		$[\lvert A\rvert]$ equals to	
(A)	4	(p)	0
(B)	6	(q)	1
(C)	8	(r)	2
(D)	12	(s)	3

49. Match the following.

Column-I		Column-II	
(A)	The number of integral solutions of the equation $(1 - i)^n = 2^n$ is	(p)	4
(B)	The number of common roots of the equations $x^3 + 2x^2 + 2x + 1 = 0$ and $x^{2000} + x^{2002} + 1 = 0$ is	(q)	3
(C)	The number of all non-zero complex numbers 'z' satisfying $\bar{z} = iz^2$ is	(r)	2
(D)	If z is a complex number, then the number of solutions of $z^2 + \lvert z\rvert = 0$ is	(s)	1

SECTION-V

Integer Answer Type

50. If $\frac{3iz_2}{5z_1}$ is purely real, then find $5\left|\frac{3z_1+7z_2}{3z_1-7z_2}\right|$.

51. If a complex number z satisfies $|z-8-4i| + |z-14-4i| = 10$, then the maximum value of $\arg(z) = \tan^{-1}\frac{11}{3k}$, find k.

52. If 'a' and 'b' are complex numbers. One of the roots of the equation $x^2 + ax + b = 0$ is purely real and the other is purely imaginary then $a^2 - \bar{a}^2 = kb$, find k.

53. The sum of the real parts of the complex numbers satisfying the equations $\left|\frac{z-4i}{z-2i}\right| = 1$ and $\left|\frac{z-8+3i}{z+3i}\right| = \frac{3}{5}$ is $\frac{k}{5}$, find k.

54. If the equation $z^4 + a_1z^3 + a_2z^2 + a_3z + a_4 = 0$ where a_1, a_2, a_3, a_4 are real coefficients different from zero has purely imaginary roots then find the value of the expression $\frac{a_3}{a_1a_2} + \frac{a_1a_4}{a_2a_3}$.

55. $\sum_{j=1}^{n-1} \frac{1}{1-e^{\frac{2\pi ij}{n}}} = \frac{n-1}{k}$, find k. $(i = \sqrt{-1})$

56. If $z_1, z_2, z_3, ..., z_n$ are in G.P with first term as unity such that $z_1 + z_2 + z_3 + ... + z_n = 0$. Now if $z_1, z_2, z_3, ..., z_n$ represents the n vertices of a polygon, then the distance between incentre and circumcentre of the polygon is represented by $4k$. Find k.

57. Let λ, z_0 be two complex numbers. $A(z_1)$, $B(z_2)$, $C(z_3)$ be the vertices of a triangle such that $z_1 = z_0 + \lambda$, $z_2 = z_0 + \lambda e^{i\pi/4}$, $z_3 = z_0 + \lambda e^{i7\pi/11}$ and $\angle ABC = \frac{3k\pi}{22}$ then the value of k is

58. The argument of $(z-a)(\bar{z}-b)$ is equal to that of $\frac{(\sqrt{3}+i)(1+\sqrt{3}i)}{1+i}$, where a, b are real numbers. If locus of z is a circle with centre $\frac{3+i}{2}$, then find $(a + b)$.

59. If $z = \frac{1}{2}(\sqrt{3} - i)$ then the least positive integral value of 'n' such that $(z^{101} + i^{109})^{106} = z^n$ is 'k', then $\frac{2}{5}k =$

SOLUTIONS

1. (b) : $x^n - 1$
$= (x - 1)(x - \alpha_1)(x - \alpha_2).....(x - \alpha_{n-1})$
$= x^n - x^{n-1}(1 + \alpha_1 + ... + \alpha_{n-1})$
$+ x^{n-2}\left(\sum_{1\le i<j\le n-1} \alpha_i\alpha_j + \alpha_1 + \alpha_2 ..... + \alpha_{n-1}\right) + ... - 1 = 0$
$\Rightarrow \sum_{1\le i<j\le n-1} \alpha_i\alpha_j + \alpha_1 + \alpha_2 ..... + \alpha_n = 0$

$$\sum_{1\le i<j\le n-1} \alpha_i \alpha_j = 1$$

2. (d) : Given that $z = \cos\theta + i\sin\theta = e^{i\theta}$

$$\therefore \sum_{m=1}^{15} \text{Im}(z^{2m-1}) = \sum_{m=1}^{15} \text{Im}(e^{i\theta})^{2m-1} = \sum_{m=1}^{15} \text{Im}\, e^{i(2m-1)\theta}$$

$= \sin\theta + \sin 3\theta + \sin 5\theta + \ldots\ldots + \sin 29\theta$

$$= \frac{\sin\left(\frac{\theta+29\theta}{2}\right)\sin\left(\frac{15\times 2\theta}{2}\right)}{\sin\left(\frac{2\theta}{2}\right)}$$

$$= \frac{\sin(15\theta)\sin(15\theta)}{\sin\theta} = \frac{1}{4\sin 2^\circ}$$

3. (b) : Let $z = a + ib$

$$\Rightarrow \quad a+ib+\sqrt{a^2+b^2} = 2+8i$$

$$\Rightarrow \quad b = 8,\ a+\sqrt{a^2+64} = 2$$

$\Rightarrow a^2 + 64 = a^2 - 4a + 4$

$\Rightarrow -4a = 60 \Rightarrow a = -15$

$$\therefore \quad |z| = \sqrt{a^2+b^2} = \sqrt{225+64} = \sqrt{289} = 17$$

4. (a) : $|3z + 9 - 7i|$

$= |3z + 6 - 3i + 3 - 4i| \le |3(z + 2 - i)| + |3 - 4i|$

$= 3|z+2-i| + \sqrt{3^2+4^2} = 3(5) + 5 = 20$

5. (d) : Let z_1, z_2 be roots of $2z^2 + 2z + \lambda = 0$

$$\Rightarrow \quad z_1+z_2 = -1,\ z_1z_2 = \frac{\lambda}{2}$$

When origin, z_1, z_2 forms equilateral triangle,
we have $z_1^2 + z_2^2 = z_1z_2$

$(z_1 + z_2)^2 = 3z_1z_2$

$$1 = \frac{3\cdot\lambda}{2} \Rightarrow \lambda = \frac{2}{3}$$

6. (c) : $\bar{z} + i\bar{w} \Rightarrow z - iw = 0 \Rightarrow z = iw$

Arg $(zw) = \pi \Rightarrow \arg(z) + \arg(w) = \pi$

$\Rightarrow \arg(iw) + \arg w = \pi \Rightarrow \arg i + 2\arg w = \pi$

$$\frac{\pi}{2} + 2\arg w = \pi \Rightarrow 2\arg w = \frac{\pi}{2}$$

$$\Rightarrow \quad \arg w = \frac{\pi}{4} \Rightarrow \arg(z) = \frac{3\pi}{4}$$

7. (d) : $|z_2 + iz_1| = |z_1| + |z_2| \Rightarrow z_2, iz_1, 0$ are collinear.

$\therefore \arg(iz_1) = \arg z_2 \Rightarrow \arg i + \arg z_1 = \arg z_2$

$$\Rightarrow \quad \arg z_2 - \arg z_1 = \frac{\pi}{2}$$

Now, $z_3 = \dfrac{z_2 - iz_1}{1-i}$

$\Rightarrow (1 - i)z_3 = z_2 - iz_1$

$\Rightarrow z_3 - z_2 = i(z_3 - z_1)$

$$\frac{z_3-z_2}{z_3-z_1} = i \Rightarrow \arg\left(\frac{z_3-z_2}{z_3-z_1}\right) = \frac{\pi}{2} \text{ and } |z_3-z_2| = |z_3-z_1|$$

$\therefore \quad AC = BC \quad \therefore AB^2 = AC^2 + BC^2$

$\Rightarrow 25 = 2AC^2 \Rightarrow \quad AC = \dfrac{5}{\sqrt{2}}$

Required area $= \dfrac{1}{2}\times\dfrac{5}{\sqrt{2}}\times\dfrac{5}{\sqrt{2}} = \dfrac{25}{4}$ sq. units

8. (a) : $A(5, -4)$, $B(-3, 2)$ subtends an angle $\dfrac{\pi}{4}$ at $C(z)$ on the circle. Hence $\dfrac{\pi}{2}$ at centre

$$OM \perp AB \ \therefore \quad AM = \frac{AB}{2} = \frac{\sqrt{64+36}}{2} = \frac{10}{2} = 5$$

$$\text{Radius} = \sqrt{25+25} = \sqrt{50} = 5\sqrt{2}$$

9. (d) : Let $x = \dfrac{3-5i}{2} \Rightarrow 2x = 3 - 5i$

$\Rightarrow (2x - 3)^2 = (-5i)^2 \Rightarrow 4x^2 - 12x + 9 = 25i^2$

$\Rightarrow 4x^2 - 12x + 34 = 0 \Rightarrow 2x^2 - 6x + 17 = 0$

$$\begin{array}{r} x+4 \\ 2x^2-6x+17\overline{)\,2x^3+2x^2-7x+72} \\ 2x^3-6x^2+17x \\ (-)\quad(+)\quad(-) \\ \hline 8x^2-24x+72 \\ 8x^2-24x+68 \\ (-)\quad(+)\quad(-) \\ \hline 4 \end{array}$$

10. (b) : Let $x = \text{cis}\,\alpha$, $y = \text{cis}\,\beta$ and $z = \text{cis}\,\gamma$

Clearly $x + y + z = 0, \dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z} = 0$

$$x^2+y^2+z^2 = (x+y+z)^2 - 2xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = 0$$

$\Rightarrow \text{cis}2\alpha + \text{cis}2\beta + \text{cis}2\gamma = 0$

$\Rightarrow \cos2\alpha + \cos2\beta + \cos2\gamma = 0$

$\Rightarrow 1 - 2\sin^2\alpha + 1 - 2\sin^2\beta + 1 - 2\sin^2\gamma = 0$

$$\Rightarrow \quad \sin^2\alpha + \sin^2\beta + \sin^2\gamma = \frac{3}{2}$$

11. (a) : We have, $z_1(z_1^2 - 3z_2^2) = 2$

$$\Rightarrow \quad z_1^2(z_1^4 + 9z_2^4 - 6z_1^2z_2^2) = 4$$

$$(z_1^2)^3 + 9z_1^2z_2^4 - 6z_1^4z_2^2 = 4 \qquad \ldots(i)$$

Also, $z_2^2(3z_1^2 - z_2^2)^2 = 121$

$$\Rightarrow \quad (z_2^2)^3 + 9z_2^2z_1^4 - 6z_1^2z_2^4 = 121 \qquad \ldots(ii)$$

(i) + (ii) $\Rightarrow (z_1^2 + z_2^2)^3 = 125 \Rightarrow z_1^2 + z_2^2 = 5$

12. (c) : Let $z = a + ib$, $b \neq 0$, where Im $z = b$

$$z^5 = (a+ib)^5 = a^5 + {}^5C_1a^4bi + {}^5C_2a^3b^2i^2 + {}^5C_3a^2b^3i^3 + {}^5C_4ab^4i^4 + i^5b^5$$

Im $z^5 = 5a^4b - 10a^2b^3 + b^5$

$$y = \frac{\text{Im}\, z^5}{\text{Im}^5 z} = 5\left(\frac{a}{b}\right)^4 - 10\left(\frac{a}{b}\right)^2 + 1$$

Let $\left(\frac{a}{b}\right)^2 = x$ (say), $x \in R^+$

$y = 5x^2 - 10x + 1 = 5[x^2 - 2x] + 1 = 5[(x-1)^2] - 4$

Hence $y_{\min} = -4$

13. (c) : We have

$$\left|\frac{z_1}{2} + \frac{z_2}{3} + \frac{z_3}{4} + \frac{z_4}{5}\right| = \frac{k}{60}\left|z_1z_2z_3z_4\right|\left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} + \frac{1}{z_4}\right|$$

Now, $z_1\bar{z}_1 = 2$, $z_2\bar{z}_2 = 3$, $z_3\bar{z}_3 = 4$ and $z_4\bar{z}_4 = 5$

$$\text{So, } k = \frac{60}{|z_1z_2z_3z_4|} = \frac{60}{\sqrt{2}\sqrt{3}\sqrt{4}\sqrt{5}} = \sqrt{30} = |z_4z_1z_2|$$

14. (b) : $\left|\frac{1}{z_2} + \frac{1}{z_1}\right| = \left|\frac{1}{z_2} - \frac{1}{z_1}\right| \Rightarrow |z_1 + z_2| = |z_1 - z_2|$

$\Rightarrow z_1\bar{z}_2 + z_2\bar{z}_1 = 0 \Rightarrow \frac{z_1}{z_2}$ is purely imaginary.

$\Rightarrow \arg\left(\frac{z_1}{z_2}\right) = \pm\frac{\pi}{2} \Rightarrow \angle POQ = \frac{\pi}{2}$

Circumcentre of ΔPOQ is the midpoint of PQ.

15. (c) : Let $A = 1 + 3\alpha + 5\alpha^2 + 7\alpha^3 + ... + 11\alpha^5 + 13\alpha^6$

$\alpha A = \alpha + 3\alpha^2 + 5\alpha^3 + 7\alpha^4 + ... + 11\alpha^6 + 13\alpha^7$

Now, $(1-\alpha)A = 1 + 2\alpha + 2\alpha^2 + 2\alpha^3 + ... + 2\alpha^6 - 13\alpha^7$

$= -12 + 2[\alpha + \alpha^2 + ... + \alpha^6] = -14 \Rightarrow A = -\frac{14}{1-\alpha}$

16. (d) : $\left|\frac{z_1+z_2}{z_1-z_2}\right| = 1 \Rightarrow \frac{z_1+z_2}{z_1-z_2} = \cos\alpha + i\sin\alpha$

where α is the argument of $\frac{z_1+z_2}{z_1-z_2}$.

Applying componendo and dividendo, we get

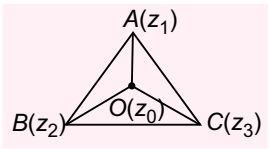
$$\frac{z_1}{z_2} = \frac{1+\cos\alpha+i\sin\alpha}{-1+\cos\alpha+i\sin\alpha}$$

$$= \frac{2\cos\left(\frac{\alpha}{2}\right)\left[\cos\left(\frac{\alpha}{2}\right)+i\sin\left(\frac{\alpha}{2}\right)\right]}{2i\sin\left(\frac{\alpha}{2}\right)\left[\cos\left(\frac{\alpha}{2}\right)+i\sin\left(\frac{\alpha}{2}\right)\right]} = -i\cot\left(\frac{\alpha}{2}\right)$$

Purely imaginary in nature.

17. (c) : Taking rotation at 'O'

$$\frac{z_0 - z_1}{z_0 - z_2} = \cos 2C - i\sin 2C$$



$$\frac{z_0 - z_3}{z_0 - z_2} = \cos 2A + i\sin 2A$$

$$\text{Now } \left(\frac{z_0 - z_1}{z_0 - z_2}\right)\frac{\sin 2A}{\sin 2B} + \left(\frac{z_0 - z_3}{z_0 - z_2}\right)\frac{\sin 2C}{\sin 2B}$$

$$= \frac{\sin 2A\cos 2C - i\sin 2A\sin 2C + \cos 2A\sin 2C + i\sin 2A\sin 2C}{\sin 2B}$$

$$= \frac{\sin(2A+2C)}{\sin 2B} = -1$$

18. (a) : $az^2 + z + 1 = 0$ ... (i)

Taking conjugate of both sides,

$\overline{az^2 + z + 1} = \bar{0} \Rightarrow \bar{a}(\bar{z})^2 + \bar{z} + \bar{1} = 0$

$\bar{a}z^2 - z + 1 = 0$ ...(ii)

(since $\bar{z} = -z$ as 'z' is purely imaginary)

Eliminating 'z' from both the equations, we get

$(\bar{a} - a)^2 + 2(a + \bar{a}) = 0$

Let $a = \cos\theta + i\sin\theta$ (since $|a| = 1$) so that

$(-2i\sin\theta)^2 + 2(2\cos\theta) = 0$

$$\Rightarrow \cos\theta = \frac{-1 \pm \sqrt{1+4}}{2}$$

Only feasible value of $\cos\theta$ is $\frac{\sqrt{5}-1}{2}$

Hence $a = \cos\theta + i\sin\theta$, where $\theta = \cos^{-1}\left(\frac{\sqrt{5}-1}{2}\right)$

19. (a) : $\cos A = \frac{1}{\sqrt{2}} \Rightarrow A = \frac{\pi}{4}$

$$\therefore \frac{z_3 - z_1}{z_2 - z_1} = \left|\frac{z_3 - z_1}{z_2 - z_1}\right| \text{cis}\,(\pi/4)$$

$$\Rightarrow \left(\frac{z_3 - z_1}{z_2 - z_1}\right)^4 = \left(\frac{\sqrt{2}}{\sqrt{3}+1}\right)^4 e^{i\pi}$$

$$\Rightarrow \left(\frac{z_3 - z_1}{z_2 - z_1}\right)^4 = -\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^4$$

20. (a) : $OP = OC - CP$

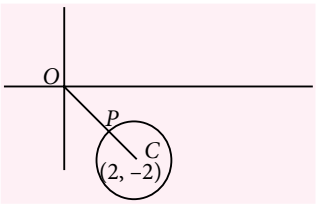
$= 2\sqrt{2} - 1$

$\therefore OP : PC = (2\sqrt{2} - 1) : 1$

coordinate of P =

$$\left(\frac{2(2\sqrt{2}-1)}{2\sqrt{2}}, \frac{-2(2\sqrt{2}-1)}{2\sqrt{2}}\right)$$

$$= \left(\left(2 - \frac{1}{\sqrt{2}}\right), -\left(2 - \frac{1}{\sqrt{2}}\right)\right)$$



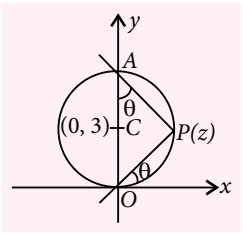
21. (a) : $(z+1)(z^2 + z + 1) \Rightarrow z = -1, \omega, \omega^2$

Let $f(z) = z^{1985} + z^{100} + 1$

$f(-1) \neq 0$, $f(\omega) = f(\omega^2) = 0 \quad \therefore \omega + \omega^2 = -1$

22. (a) : $r = OA\ \sin\theta = 6\sin\theta$

$z = 6\sin\theta\ (\cos\theta + i\sin\theta) \Rightarrow \cot\theta - \frac{6}{z} = i$



23. (d) : If $z_1 + \sqrt{z_1^2 - z_2^2} = u$ and $z_1 - \sqrt{z_1^2 - z_2^2} = v$

We have

$$|u|^2 + |v|^2 = \frac{1}{2}|u+v|^2 + \frac{1}{2}|u-v|^2 = 2|z_1|^2 + 2\left|z_1^2 - z_2^2\right|$$

And so $\left(|u| + |v|\right)^2 = 2\left\{|z_1|^2 + \left|z_1^2 - z_2^2\right| + |z_2|^2\right\}$

$$= |z_1 + z_2|^2 + |z_1 - z_2|^2 + 2\left|z_1^2 - z_2^2\right|$$

$$= \left(|z_1 + z_2| + |z_1 - z_2|\right)^2$$

24. (a) : Let $z_1 = \cos\phi_1 + i\sin\phi_1$ and $z_2 = \cos\phi_2 + i\sin\phi_2$ be the roots of $z^2 + az + b = 0$

$z_1 + z_2 = (-a)$ and $z_1 z_2 = b$

$$-2\cos\left(\frac{\phi_1 - \phi_2}{2}\right)\left[\cos\left(\frac{\phi_1 + \phi_2}{2}\right) + i\sin\left(\frac{\phi_1 + \phi_2}{2}\right)\right] = a$$

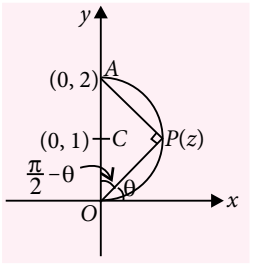
$$\Rightarrow \quad \arg a = \frac{\phi_1 + \phi_2}{2}$$

$\arg b = \phi_1 + \phi_2$

$\therefore$ $\arg b = 2\ \arg a$

Also $|z_1 z_2| = |b| = 1$ and $|a| \le 2$

25. (c) : $\angle AOP = \frac{\pi}{2} - \theta$



$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta = \frac{|z|}{2}$$

Also $\frac{2i}{z} = \frac{OA}{OP}\left(\sin\theta + i\cos\theta\right)$

$$= \frac{2}{|z|}\left(\sin\theta + i\cos\theta\right)$$

$= 1 + i\cot\theta \quad \Rightarrow \frac{2}{z} = -i + \cot\theta \quad \Rightarrow \quad \cot\theta - \frac{2}{z} = i$

26. (a, b, c) : $|z_1| = |z_2| = 1$

$\Rightarrow a^2 + b^2 = c^2 + d^2 = 1$ ...(i)

And $\mathrm{Re}(z_1\bar{z}_2) = 0 \Rightarrow \mathrm{Re}\{(a+ib)(c-id)\} = 0$

$\Rightarrow \quad ac + bd = 0$ ...(ii)

Now from (i) and (ii), we get

$$a^2 + b^2 = 1 \Rightarrow \quad a^2 + \frac{a^2c^2}{d^2} = 1 \Rightarrow \quad a^2 = d^2 \qquad \text{...(iii)}$$

$$\text{Also } c^2 + d^2 = 1 \Rightarrow \quad c^2 + \frac{a^2c^2}{b^2} = 1 \Rightarrow \quad b^2 = c^2 \qquad \text{...(iv)}$$

$|\omega_1| = \sqrt{a^2 + c^2} = \sqrt{a^2 + b^2} = 1$ [From (i) and (iv)]

and $|\omega_2| = \sqrt{b^2 + d^2} = \sqrt{c^2 + d^2} = 1$

[From (i) and (iv)]

Further $\mathrm{Re}(\omega_1\bar{\omega}_2) = \mathrm{Re}\{(a+ic)(b-id)\} = ab + cd = 0$

[From (ii) and (iv)]

Also $\mathrm{Im}(\omega_1\bar{\omega}_2) = bc - ad = \pm 1$

$\left[\because\ |\omega_1| = 1, |\omega_2| = 1 \text{ and } \mathrm{Re}(\omega_1\bar{\omega}_2) = 0\right]$

27. (a, b, c) : $|z_1 + z_2| = |z_1 - z_2|$

$\Rightarrow \quad (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) = (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$

$$\Rightarrow \quad z_1\bar{z}_2 + z_2\bar{z}_1 = 0 \Rightarrow \quad \frac{z_1}{z_2} = -\left(\frac{\bar{z}_1}{\bar{z}_2}\right)$$

$\Rightarrow \frac{z_1}{z_2}$ is purely imaginary.

Also, $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2$

ΔOAB is a right angled triangle, right angled at O.

So, circumcentre $= \frac{z_1 + z_2}{2}$

28. (a) : $z = (1+\sqrt{2}) + i,\ \bar{z} = (1+\sqrt{2}) - i$

For adjacent vertices

$$z = \bar{z}\,\mathrm{cis}\frac{2\pi}{n} \Rightarrow \mathrm{cis}\frac{2\pi}{n} = \frac{1 + i(\sqrt{2}-1)}{1 - i(\sqrt{2}-1)} = \frac{1 + i\tan\frac{\pi}{8}}{1 - i\tan\frac{\pi}{8}} = \mathrm{cis}\frac{2\pi}{8}$$

$\Rightarrow \quad n = 8$

29. (a, b, d) : Let $|\alpha| = r > 1$ and $\alpha = r\mathrm{cis}\theta$ then

$$z = x + iy = \alpha + \frac{1}{\alpha} = r\mathrm{cis}\theta + \frac{\mathrm{cis}(-\theta)}{r}$$

$$\Rightarrow \quad x = \left(r + \frac{1}{r}\right)\cos\theta \text{ and } y = \left(r - \frac{1}{r}\right)\sin\theta$$

Eliminating θ gives $\dfrac{x^2}{\left(r + \frac{1}{r}\right)^2} + \dfrac{y^2}{\left(r - \frac{1}{r}\right)^2} = 1,$

which is an ellipse.

$a = r + \frac{1}{r},\ b = r - \frac{1}{r}\ (r = |\alpha| > 1 \Rightarrow a > b)$

$\therefore \quad e = \sqrt{1 - \frac{b^2}{a^2}} = \frac{2}{r + \frac{1}{r}}$, distance between foci $= 2ae = 4$

Distance between directrix $= \frac{2a}{e}$

30. (b, d) : $z = 2\cos\frac{3\pi}{5}\left(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right)$

$$= 2\cos\frac{2\pi}{5}\left(\cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}\right)$$

31. (b, c) : $z = (z_1 + z_2 + \ldots + z_n)\left(\frac{1}{z_1} + \frac{1}{z_2} + \ldots + \frac{1}{z_n}\right)$

$\Rightarrow |z_1 + z_2 + ... + z_n|^2 \rightarrow$ which is real

$\le |z_1|^2 + |z_2|^2 + |z_3|^2 + ... + |z_n|^2 = n^2$

32. (a, d) : $|a| = |b| = |c| = r$

$|c| = |-az^2 - bz| \le r|z|^2 + r|z|$

$\Rightarrow |z|^2 + |z| - 1 \ge 0$ ...(i)

and $az^2 = -(bz + c)$

$|a||z|^2 \le r|z| + r$

$\Rightarrow |z|^2 - |z| - 1 \le 0$ ...(ii)

Solve (i) and (ii) to conclude that

$$\min|z| = \frac{\sqrt{5}-1}{2} \text{ and } \max|z| = \frac{\sqrt{5}+1}{2}$$

33. (a,d) : If $|z - 1| \le |z - 5|$

Then $|z|^2 - z - \bar{z} + 1 \le |z|^2 - 5z - 5\bar{z} + 25 \,\&\, |z-3| = |z-1|$

$\Rightarrow \quad 4(z + \bar{z}) \le 24$ and $z\bar{z} - 3z - 3\bar{z} + 9 = z\bar{z} - z - \bar{z} + 1$

$\Rightarrow \quad z + \bar{z} \le 6$ and $2(z + \bar{z}) = 8 \Rightarrow z + \bar{z} = 4 \Rightarrow \text{Re}(z) = 2$

$\Rightarrow \quad \text{Re}(z) \le 3$

Similarly, for $|z - 1| \ge |z - 5|$, we get $\text{Re}(z) = 4$

34. (a, d) : We observe that if z_0 is a non-zero complex number and c is a real number, then the equation $\bar{z}_0 z + z_0 \bar{z} + c = 0$ represents a straight line with complex slope $\frac{-z_0}{\bar{z}_0}$.

Let $L_1 : \bar{\alpha}z + \alpha\bar{z} + c = 0$ and $L_2 : \bar{\beta}z + \beta\bar{z} + d = 0$

where $\alpha = (a, b)$ and $\beta = (p, q)$ are non zero complex numbers. Then their cartesian equations are

$$ax + by + \frac{c}{2} = 0 \text{ and } px + qy + \frac{d}{2} = 0$$

$\therefore \quad L_1 \perp L_2 \Leftrightarrow ap + bq = 0 \Rightarrow \alpha\bar{\beta} + \bar{\alpha}\beta = 0$

$$\Leftrightarrow \frac{\alpha}{\alpha} + \frac{\beta}{\beta} = 0 \Leftrightarrow \mu_1 + \mu_2 = 0,$$

where $\mu_1 = -\frac{\alpha}{\alpha}$ and $\mu_2 = \frac{-\beta}{\beta}$ are the complex slopes of L_1 and L_2 respectively.

$L_1 \parallel L_2 \Leftrightarrow aq - bp = 0 \Leftrightarrow \alpha\bar{\beta} - \bar{\alpha}\beta = 0$

$$\Leftrightarrow \frac{\alpha}{\alpha} = \frac{\beta}{\beta} \Leftrightarrow \mu_1 = \mu_2$$

35. (a) : $z_1 = \cos\theta + i\sin\theta$

$z_2 = \cos2\theta + i\sin2\theta - (\cos\theta + i\sin\theta)$

$= (\cos2\theta - \cos\theta) + i(\sin2\theta - \sin\theta)$

$|z_2|^2 = (\cos2\theta - \cos\theta)^2 + (\sin2\theta - \sin\theta)^2$

$= 2 - 2(\cos2\theta\cos\theta + \sin2\theta\sin\theta) = 2 - 2\cos\theta = 4\sin^2\frac{\theta}{2}$

36. (c) : $z_2 = (\cos2\theta - \cos\theta) + i(\sin2\theta - \sin\theta)$

$$= -2\sin\frac{3\theta}{2}\sin\frac{\theta}{2} + i\,2\cos\frac{3\theta}{2}\sin\frac{\theta}{2}$$

$$= 2i\sin\frac{\theta}{2}\left(\cos\frac{3\theta}{2} + i\sin\frac{3\theta}{2}\right)$$

$$\arg z_2 = \arg\left(2i\sin\frac{\theta}{2}\right) + \arg\left(\cos\frac{3\theta}{2} + i\sin\frac{3\theta}{2}\right)$$

$$= \frac{\pi}{2} + \frac{3\theta}{2}$$

$$4n\pi < \theta < (4n+2)\pi \Rightarrow 2n\pi < \frac{\theta}{2} < (2n+1)\pi$$

$$\Rightarrow \quad \sin\frac{\theta}{2} > 0$$

37. (b) : $(4n + 2)\pi < \theta < (4n + 4)\pi$

$$\Rightarrow \quad (2n+1)\pi < \frac{\theta}{2} < (2n+2)\pi \quad \Rightarrow \quad \sin\frac{\theta}{2} < 0$$

$$\therefore \quad \arg z_2 = \frac{3\pi}{2} + \frac{3\theta}{2}$$

38. (b) : $\left(2 - 2\cos\frac{2\pi}{n}\right) + \left(2 - 2\cos\frac{4\pi}{n}\right)$

$$+ ..... + \left(2 - 2\cos(n-1)\frac{2\pi}{n}\right)$$

$$= 2(n-1) - 2\left(\cos\frac{2\pi}{n} + \cos\frac{4\pi}{n} + ... + \cos\frac{(n-1)2\pi}{n}\right)$$

$= 2(n - 1) - 2(-1) = 2n$

39. (b) : Consider the triangle OA_1A_j

Let angle $A_1OA_j = \alpha$

Then $(A_1A_j)^2 = 1^2 + 1^2 - 2(1)(1)\cos\alpha$

$$= 2(1 - \cos\alpha) = 2 \cdot 2\sin^2\frac{\alpha}{2}$$

$$\Rightarrow \quad A_1A_j = 2\sin\frac{\alpha}{2} \text{ but } a = \frac{2\pi}{n}(j-1)$$

$$\Rightarrow \quad A_1A_j = 2\sin\frac{(j-1)\pi}{n}$$

40. (b) : $2^{n-1}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}.....\sin\frac{(n-1)\pi}{n} = n$

41. (b) : $w = \frac{z-i}{2z+1} = \frac{(2x^2 + x + 2y^2 - 2y) + i(y - 2x - 1)}{(2x+1)^2 + 4y^2}$

$$\text{Re}(w) = 0 \Rightarrow x^2 + y^2 + \frac{x}{2} - y = 0$$

42. (c) : $\text{Im}(w) = 0 \quad \Rightarrow \quad y = 2x + 1$

43. (c) : $|w| = 1 \Rightarrow w\bar{w} = 1 \Rightarrow x^2 + y^2 + \frac{4x}{3} + \frac{2y}{3} = 0$

44. (a) : Locus of 'z' is perpendicular bisector of the line segment joining (0, 5) and (0, –5). Its equation is $y = 0$ (*i.e.*, x-axis)

45. (b) : Let $w = -i + \frac{15}{z} \Rightarrow w + i = \frac{15}{z}$

$$\Rightarrow \quad |w + i| = \frac{15}{|z|} = 3$$

Locus of 'w' is a circle with centre (0, –1) and radius 3.

46. (c) : $|z - i| \le 2$ represents a disc with centre at (0, 1) and radius 2.

$|iz + z_1| = |i|\,|z - iz_1| = |z - i(3 + 4i)|$

$= |(z - i) + (4 - 2i)| \le |z - i| + |4 - 2i| \le 2 + \sqrt{20}$

Maximum value of $|iz + z_1| = \sqrt{20} + 2$

47. A → r; B → p; C → s; D → q

(A) $1 + z^3 = 1 + \cos 3\alpha + i\sin 3\alpha$

$$= 2\cos\left(\frac{3\alpha}{2}\right)\left[\cos\frac{3\alpha}{2} + i\sin\frac{3\alpha}{2}\right]$$

$$\Rightarrow \quad \arg(1 + z^3) = \frac{3\alpha}{2}$$

(B) $1 - z^4 = 1 - \cos 4\alpha - i\sin 4\alpha$

$$= 2\sin 2\alpha\left[\cos\left(2\alpha - \frac{\pi}{2}\right) + i\sin\left(2\alpha - \frac{\pi}{2}\right)\right]$$

$$\Rightarrow \quad \arg(1 - z^4) = 2\alpha - \frac{\pi}{2}$$

(C) $$\frac{1 + z^3}{1 - z^4} = \frac{\cos\frac{3\alpha}{2}}{\sin 2\alpha}\left[\cos\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) + i\sin\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)\right]$$

$$\Rightarrow \quad \arg\left(\frac{1 + z^3}{1 - z^4}\right) = \frac{\pi}{2} - \frac{\alpha}{2}$$

(D) $$\frac{z^4 - 1}{z^3 + 1} = \frac{\sin 2\alpha}{\cos\left(\frac{3\alpha}{2}\right)}\left[\cos\left(\frac{\alpha}{2} + \frac{\pi}{2}\right) + i\sin\left(\frac{\alpha}{2} + \frac{\pi}{2}\right)\right]$$

$$\arg\left(\frac{z^4 - 1}{z^3 + 1}\right) = \frac{\pi}{2} + \frac{\alpha}{2}$$

48. A → r; B → r; C → q; D → q

Let α be an interior angle of the polygon

$z_n - z_1 = (z_2 - z_1)e^{i\alpha} \Rightarrow z_n = (1 - e^{i\alpha})z_1 + e^{i\alpha}z_2$

$z_3 - z_2 = (z_1 - z_2)e^{-i\alpha} \Rightarrow z_3 = e^{-i\alpha}z_1 + (1 - e^{-i\alpha})z_2$

$z_3 + z_n = (1 - 2i\sin\alpha)\,z_1 + (1 + 2i\sin\alpha)z_2$

$A = 1 - 2i\sin\alpha$

$$|A| = \sqrt{3 - 2\cos 2\alpha} = \sqrt{3 - 2\cos\left(\frac{4\pi}{n}\right)}$$

When $n = 4 \Rightarrow |A| = \sqrt{5}$

$n = 6 \Rightarrow |A| = 2$

$n = 8 \Rightarrow |A| = \sqrt{3}$

$n = 12 \Rightarrow |A| = \sqrt{2}$

49. A → s; B → r; C → q; D → q

(A) $|1 - i|^n = 2^n \Rightarrow n/2 = 0 \Rightarrow n = 0$

(B) $x^3 + 2x^2 + 2x + 1 = 0 \Rightarrow x = -1, \omega, \omega^2$

But $x = \omega, \omega^2$ will only satisfy $x^{2000} + x^{2002} + 1 = 0$

(C) $x + 2xy = 0$ and $x^2 - y^2 + y = 0$

$$\Rightarrow \quad i,\ \frac{\sqrt{3}}{2} - \frac{i}{2},\ -\frac{\sqrt{3}}{2} - \frac{i}{2}$$

(D) $x^2 - y^2 + \sqrt{x^2 + y^2} = 0$ and $2xy = 0 \Rightarrow z = 0, i, -i$

50. (5) : Let $\frac{3iz_2}{5z_1} = K$ (real) then $\frac{z_2}{z_1} = \frac{5K}{3i}$

$$5\left|\frac{3 + 7\frac{z_2}{z_1}}{3 - 7\frac{z_2}{z_1}}\right| = 5\left|\frac{3 + \frac{35K}{3i}}{3 - \frac{35K}{3i}}\right| = 5\left|\frac{35K + 9i}{35K - 9i}\right| = 5$$

51. (4) : Locus of z is an ellipse

$$\frac{(x - 11)^2}{25} + \frac{(y - 4)^2}{16} = 1$$

Equation of tangent is $y - 4 = m(x - 11) + c$

$\Rightarrow c = 11m - 4$

As $c^2 = a^2m^2 + b^2$ for standard ellipse

$$\Rightarrow \quad (11m - 4)^2 = 25m^2 + 16 \Rightarrow m = 0 \text{ or } m = \frac{11}{12}$$

$$\therefore \quad \tan\theta = \frac{11}{12} \Rightarrow \theta = \tan^{-1}\frac{11}{12}$$

52. (4) : Let α and $i\beta$, $\alpha, \beta \in R$ are roots of $x^2 + ax + b = 0 \Rightarrow \alpha + i\beta = -a,\ i\alpha\beta = b$

$\alpha - i\beta = -\bar{a}$

$\Rightarrow \quad 2\alpha = -(a + \bar{a})$ and $2i\beta = -(a - \bar{a})$

$\therefore \quad 4i\alpha\beta = a^2 - \bar{a}^2 \Rightarrow 4b = a^2 - \bar{a}^2$

53. (5) : $\left|\frac{z - 4i}{z - 2i}\right| = 1 \Rightarrow z = x + 3i$, using this in

$$\left|\frac{z - 8 + 3i}{z + 3i}\right| = \frac{3}{5} \Rightarrow 5|x - 8 + 6i| = 3|x + 6i|$$

$\Rightarrow x = 8, 17$

Two complex numbers are $8 + 3i$, $17 + 3i$

Sum of real parts = 8 + 17 =25

54. (1) : Let $z = iy$

$$\Rightarrow \quad y^4 - a_1y^3i - a_2y^2 + ia_3y + a_4 = 0$$

$$\Rightarrow \quad y^4 - a_2y^2 + a_4 = 0 \quad \text{....(i) and} \quad -a_1y^3 + a_3y = 0$$

$$\Rightarrow \quad y = 0 \text{ or } y^2 = \frac{a_3}{a_1} \quad \text{....(ii)}$$

From (i) and (ii), we get

$$\frac{a_3^2}{a_1^2} - \frac{a_2a_3}{a_1} + a_4 = 0 \Rightarrow \frac{a_3}{a_1a_2} + \frac{a_1a_4}{a_2a_3} = 1$$

55. (2) : Let $e^{i\frac{2\pi}{n}} = \alpha$ then

$$\sum_{j=1}^{n-1}\frac{1}{1 - e^{\frac{2\pi}{n}ij}} = \frac{1}{1 - \alpha} + \frac{1}{1 - \alpha^2} + \ldots\ldots + \frac{1}{1 - \alpha^{n-1}}$$

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where α is a n^{th} root of unity. $(\alpha, \alpha^2, \alpha^3, ..., \alpha^{n-1})$ are the roots of $\frac{x^n-1}{x-1} = (x-\alpha)(x-\alpha^2).....(x-\alpha^{n-1})$

Taking log on both sides

$$\log\frac{x^n-1}{x-1} = \log(x-\alpha)+\log(x-\alpha^2)+.....+\log(x-\alpha^{n-1})$$

Differentiating w.r.t. x and use $\lim_{x\to 1}$

$$\Rightarrow \quad \frac{n-1}{2} = \frac{1}{1-\alpha}+\frac{1}{1-\alpha^2}+.....+\frac{1}{1-\alpha^{n-1}}$$

56. (0) : Let vertices be $1, \alpha, \alpha^2, ..., \alpha^{n-1}$

Given $1 + \alpha + \alpha^2 + ... \alpha^{n-1} = 0 \Rightarrow \alpha^n - 1 = 0$

$\Rightarrow z_1, z_2, z_3, ..., z_n$ are roots of $\alpha^n = 1$

which form regular polygon. So, distance is zero.

57. (5) : $|z_1 - z_0| = |z_2 - z_0| = |z_3 - z_0| = |\lambda|$

$$\frac{z_3-z_0}{z_2-z_0} = \frac{e^{i7\pi/11}}{e^{i\pi/4}} = e^{i17\pi/44}$$

$$\Rightarrow \quad \angle BSC = \frac{17\pi}{44} \Rightarrow \angle BAC = \frac{17\pi}{88}$$

Similarly $\frac{z_2-z_0}{z_1-z_0} = e^{i\pi/4} \Rightarrow \angle ACB = \frac{\pi}{8}$

$$\therefore \quad \angle ABC = \pi - \frac{\pi}{8} - \frac{17\pi}{88} = \frac{15\pi}{22}$$

58. (3) : $\tan^{-1}\frac{(a-b)y}{x^2+y^2-(a+b)x+ab} = \frac{\pi}{4}$

$\Rightarrow x^2 + y^2 - (a + b)x - (a - b)y + ab = 0$

Centre $= \frac{3+i}{2} \Rightarrow a+b=3$

59. (4) : $z = \frac{-1}{2}i(1+i\sqrt{3}) = i\omega^2$

$z^{101} = i\omega$

$(z^{101} + i^{109})^{106} = (-i\omega^2)^{106} = -\omega^2$

$-\omega^2 = (i\omega^2)^n = i^n\omega^{2n}$ or $\omega^{2n-2}\, i^n = -1$

This is possible only when $n = 4r + 2$ and $2n - 2$ is multiple of 3 *i.e.*,

$2(4r + 2) - 2$ is a multiple of 3.

i.e., $8r + 2$ is a multiple of $3 \Rightarrow r = 2$

$\therefore \quad n = 10$

$\therefore \quad \frac{2}{5}k = 4$

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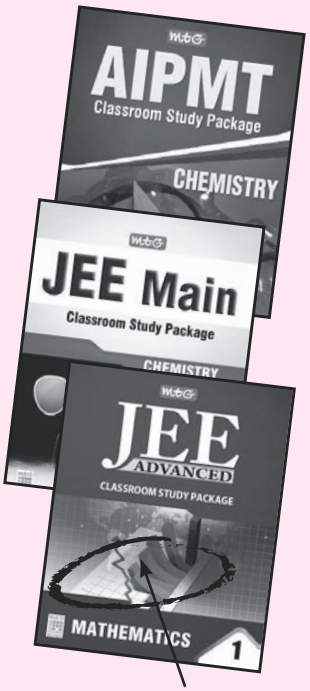

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CONCEPT BOOSTERS

Class XII

MATRICES AND DETERMINANTS

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This column is aimed at Class XII students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

DEFINITION OF MATRIX

A system of $m \times n$ numbers arranged in the form of an ordered set of m rows and n columns is called an $m \times n$ matrix. It can be read as m by n matrix. It is represented as $A = [a_{ij}]_{m \times n}$ and can be written in expanded form as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

DIFFERENT TYPES OF MATRICES

(i) Rectangular matrix and square matrix : If in an $m \times n$ matrix $m \neq n$, then it is a rectangular matrix

But if $m = n$, it is a square matrix.

(ii) Column Matrix : A matrix which has only one column and m rows is called a column matrix of length m.

(iii) Row Matrix : A matrix which has only one row and n columns is called a row matrix of length n.

(iv) Diagonal Matrix : A square matrix of any order with zero elements everywhere, except on the main diagonal, is called a diagonal matrix.

(v) Scalar matrix : A matrix whose diagonal elements are all equal and other entries are zero, is called a scalar matrix.

(vi) Identity or Unit Matrix : A square matrix in which all the elements along the main diagonal (elements of the form a_{ii}) are unity is called an identity matrix or a unit matrix. An identity matrix of order n is denoted by I_n.

(vii) Null or Zero Matrix : The matrix whose all elements are zero is called null matrix or zero matrix. It is usually denoted by O.

(viii) Triangular Matrix : A square matrix whose elements above the main diagonal or below the main diagonal are all zero is called a triangular matrix.

Note : **(i)** $[a_{ij}]_{n \times n}$ is said to be upper triangular matrix if $i > j \Rightarrow a_{ij} = 0$,

(ii) $[a_{ij}]_{n \times n}$ is said to be lower triangular matrix if $i < j \Rightarrow a_{ij} = 0$.

(ix) Sub Matrix : A matrix obtained by omitting some rows or some columns or both of a given matrix A is called a sub matrix of A.

(x) Horizontal Matrix : Any matrix in which the number of columns is more than the number of rows is called a horizontal matrix.

(xi) Vertical Matrix : Any matrix in which the number of rows is more than the number of columns is called vertical matrix.

EQUALITY OF TWO MATRICES

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if they are of the same order and their corresponding elements are equal. If two matrices A and B are equal, we write $A = B$.

OPERATION ON MATRICES

Addition of Matrices

If A and B are two matrices of the same order $m \times n$, then their sum is defined to be the matrix of order $m \times n$ obtained by adding the corresponding elements of A and B.

---

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Subtraction of Two Matrices

If A and B are two matrices of the same order then the sum $A + (-B)$, written as $A - B$, is the matrix obtained by subtracting B from A.

Properties of Matrix Addition

If A, B, C are three matrices of the same order, then

(i) $A + B = B + A$ (commutative law of addition)

(ii) $A + (B + C) = (A + B) + C$ (associative law of addition)

(iii) $A + O = O + A = A$

(iv) $A + B = A + C \Rightarrow B = C$

(v) $A + (-A) = (-A) + A = O$.

Multiplication of a Matrix by a Scalar

If $A = [a_{ij}]_{m \times n}$ and λ is a scalar,

then $\lambda A = [\lambda a_{ij}]_{m \times n}$

Multiplication of Two Matrices

For two matrices A and B, the product matrix AB can be obtained if the number of columns in A = the number of rows in B.

Properties of Matrix Multiplication

(i) Matrix multiplication is associative

i.e., $(AB)C = A(BC)$, A, B and C are $m \times n$, $n \times p$ and $p \times q$ matrices respectively.

(ii) Multiplication of matrices is distributive over addition of matrices *i.e.*, $A(B + C) = AB + AC$

(iii) Existence of multiplicative identity of square matrices. If A is a square matrix of order n and I_n is the identity matrix of order n, then $AI_n = I_nA = A$.

(iv) Whenever AB and BA both exist, it is not necessary that $AB = BA$.

(v) The product of two matrices can be a zero matrix while neither of them is a zero matrix.

(vi) In the case of matrix multiplication of $AB = 0$, then it doesn't necessarily imply that $A = 0$ or $B = 0$ or $BA = 0$.

TRACE OF A MATRIX

Let A be a square matrix of order n. The sum of the diagonal elements of A is called the trace of A.

$$\therefore \quad \text{Trace}\,(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \ldots + a_{nn}.$$

TRANSPOSE OF A MATRIX

The matrix obtained from any given matrix A, by interchanging rows and columns, is called the transpose of A and is denoted by A' or A^T.

Properties of Transpose of a matrix

(i) $(A')' = A$

(ii) $(A + B)' = A' + B'$

(iii) $(\alpha A)' = \alpha A'$, α being any scalar.

(iv) $(AB)' = B'A'$

SPECIAL MATRICES

- **Symmetric Matrix**

  A matrix which is unchanged by transposition is called a symmetric matrix. Such a matrix is necessarily square *i.e.*, if $A = [a_{ij}]_{m \times n}$ is a symmetric matrix then $m = n$, $a_{ij} = a_{ji}$ *i.e.*, $A' = A$.

- **Skew Symmetric Matrix**

  A square matrix $A = [a_{ij}]$ is said to be skew symmetric, if $a_{ij} = -a_{ji}$ for all i and j.

  Thus if $A = [a_{ij}]_{m \times n}$ is a skew symmetric matrix, then $m = n$, $a_{ij} = -a_{ji}$ *i.e.*, $A' = -A$.

  Obviously diagonal elements of a square matrix are zero.

  Note : Every square matrix can be uniquely expressed as the sum of symmetric and skew symmetric matrix.

  i.e., $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$

  where $\frac{1}{2}(A + A')$ and $\frac{1}{2}(A - A')$

  are symmetric and skew symmetric parts of A.

- **Orthogonal Matrix**

  A square matrix A is said to be orthogonal, if $AA' = A'A = I$, where I is a unit matrix.

  Note :

  (i) If A is orthogonal, then A' is also orthogonal.

  (ii) If A and B are orthogonal matrices then AB and BA are also orthogonal matrices.

- **Idempotent Matrix :** A square matrix A is called idempotent provided if it satisfies the relation $A^2 = A$.

- **Involutory Matrix**

  A square matrix A is called involutory matrix if $A^2 = I$

- **Periodic Matrix**

  A square matrix A is called periodic, if $A^{k+1} = A$, where k is a positive integer. If k is the least positive integer for which $A^{k+1} = A$, then k is said to be period of A. For $k = 1$, we get $A^2 = A$ and we called it to be idempotent matrix.

- **Nilpotent Matrix**

  A square matrix A is called a nilpotent matrix, if there exists a positive integer m such that $A^m = O$. If m is the least positive integer such that $A^m = O$, then m is called the index of the nilpotent matrix A or nilpotency of matrix.

DETERMINANT

Equations $a_1x + b_1y = 0$ and $a_2x + b_2y = 0$ in x and y have a unique solution if and only if $a_1b_2 - a_2b_1 \neq 0$. We write $a_1b_2 - a_2b_1$ as $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ and call it a determinant of order 2.

Similarly the equations $a_1x + b_1y + c_1z = 0$, $a_2x + b_2y + c_2z = 0$ and $a_3x + b_3y + c_3z = 0$ have a unique solution if $a_1(b_2c_3 - b_3c_2) + b_1(a_3c_2 - a_2c_3) + c_1(a_2b_3 - a_3b_2) \neq 0$

$$i.e., \quad \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$$

The number a_i, b_i, c_i $(i = 1, 2, 3)$ are called the elements of the determinant.

The determinant obtained by deleting the i^{th} row and j^{th} column is called the minor of the element at the i^{th} row and j^{th} column. We shall denote it by M_{ij}. The cofactor of this element is $(-1)^{i+j} M_{ij}$, denoted by C_{ij}.

Let $A = [a_{ij}]_{3\times 3}$ be a matrix, then the corresponding determinant (denoted by det A or $|A|$) is

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

It is easy to see that $|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$ (we say that we have expanded the determinant $|A|$ along first row). Infact value of $|A|$ can be obtained by expanding it along any row or along any column. Further note that if elements of a row (column) are multiplied to the cofactors of other row (column) and then added, then the result is zero : $a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23} = 0$.

PROPERTIES OF DETERMINANTS

- The value of a determinant remains unaltered, if its rows are changed into columns and the columns into rows.
- If all the elements of a row (or column) of a determinant are zero, then the value of the determinant is zero.
- If any two rows (or columns) of a determinant are identical, then the value of the determinant is zero.
- The interchange of any two rows (columns) of a determinant results in change of its sign.
- If all the elements of a row (column) of a determinant are multiplied by a non-zero constant, then the determinant gets multiplied by that constant.
- If each element of a row (or column) of a determinant is a sum of two terms, then determinant can be written as sum of two determinant in the following way:

$$\begin{vmatrix} a_1 & b_1 & c_1 + d_1 \\ a_2 & b_2 & c_2 + d_2 \\ a_3 & b_3 & c_3 + d_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

- The value of a determinant remains unaltered under a column operation of the form $C_i \to C_i + \alpha C_j + \beta C_k (j, k \neq i)$ or a row operation of the form $R_i \to R_i + \alpha R_j + \beta R_k$ $(j, k \neq i)$

Product of two determinants

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1l_1 + b_1m_1 + c_1n_1 & a_1l_2 + b_1m_2 + c_1n_2 & a_1l_3 + b_1m_3 + c_1n_3 \\ a_2l_1 + b_2m_1 + c_2n_1 & a_2l_2 + b_2m_2 + c_2n_2 & a_2l_3 + b_2m_3 + c_2n_3 \\ a_3l_1 + b_3m_1 + c_3n_1 & a_3l_2 + b_3m_2 + c_3n_2 & a_3l_3 + b_3m_3 + c_3n_3 \end{vmatrix}$$

(row by column multiplication)

$$= \begin{vmatrix} a_1l_1 + b_1l_2 + c_1l_3 & a_1m_1 + b_1m_2 + c_1m_3 & a_1n_1 + b_1n_2 + c_1n_3 \\ a_2l_1 + b_2l_2 + c_2l_3 & a_2m_1 + b_2m_2 + c_2m_3 & a_2n_1 + b_2n_2 + c_2n_3 \\ a_3l_1 + b_3l_2 + c_3l_3 & a_3m_1 + b_3m_2 + c_3m_3 & a_3n_1 + b_3n_2 + c_3n_3 \end{vmatrix}$$

(row by row multiplication)

We can also multiply determinants column by row or column by column.

Limit of a determinant

$$\text{Let } \Delta(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix},$$

$$\text{then } \lim_{x\to a} \Delta(x) = \begin{vmatrix} \lim_{x\to a} f(x) & \lim_{x\to a} g(x) & \lim_{x\to a} h(x) \\ \lim_{x\to a} l(x) & \lim_{x\to a} m(x) & \lim_{x\to a} n(x) \\ \lim_{x\to a} u(x) & \lim_{x\to a} v(x) & \lim_{x\to a} w(x) \end{vmatrix},$$

provided each of nine limiting values exist finitely.

Differentiation of a determinant

$$\text{Let } \Delta(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix},$$

$$\text{then } \Delta'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$$

$$+ \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

Integration of a Determinant

Let $\Delta(x)=\begin{vmatrix} f(x) & g(x) & h(x) \\ a & b & c \\ l & m & n \end{vmatrix}$,

where a, b, c, l, m and n are constants,

then $\int_a^b \Delta(x)\,dx=\begin{vmatrix} \int_a^b f(x)dx & \int_a^b g(x)dx & \int_a^b h(x)\,dx \\ a & b & c \\ l & m & n \end{vmatrix}$

Note that if more than one row (column) of $\Delta(x)$ are variable, then in order to find $\int_a^b \Delta(x)\,dx$. First we evaluate the determinant $\Delta(x)$ by using the properties of determinants and then we integrate it.

SPECIAL DETERMINANTS

Skew symmetric Determinant

A determinant of a skew symmetric matrix of odd order is always zero.

Circulant Determinant

A determinant is called circulant if its rows (columns) are cyclic shifts of the first row (columns).

e.g., $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$. It can be shown that its value is $-(a^3+b^3+c^3-3abc)$.

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}=(a-b)(b-c)(c-a)$$

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}=(a-b)(b-c)(c-a)(a+b+c)$$

$$\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}=(a-b)(b-c)(c-a)(ab+bc+ca)$$

ADJOINT OF A SQUARE MATRIX

Let $A=[a_{ij}]_{n\times n}$ be an $n\times n$ matrix. The transpose B' of the matrix $B=[A_{ij}]_{n\times n}$, where A_{ij} denotes the cofactor of the elements a_{ij} in the $|A|$, is called the adjoint of the matrix A and is denoted by the symbol adj A.

Thus, the adjoint of a matrix A is the transpose of the matrix formed by the cofactors of A.

Properties of Adjoint Matrix

If A, B are square matrices of order n and I_n is corresponding unit matrix, then

(i) $A(\text{adj } A) = |A|\, I_n = (\text{adj } A)\, A$
(Thus $A(\text{adj } A)$ is always a scalar matrix)

(ii) $|\text{adj } A| = |A|^{n-1}$

(iii) $\text{adj } (\text{adj } A) = |A|^{n-2}\, A$

(iv) $|\text{adj } (\text{adj } A)| = |A|^{(n-1)^2}$

(v) $\text{adj } (A^T) = (\text{adj } A)^T$

(vi) $\text{adj } (AB) = (\text{adj } B)\cdot(\text{adj } A)$

(vii) $\text{adj } (A^m) = (\text{adj } A)^m,\ m \in N$

(viii) $\text{adj } (kA) = k^{n-1}\ (\text{adj } A),\ k \in R$

(ix) $\text{adj } (I_n) = I_n$

(x) $\text{adj } 0 = 0$

A is symmetric $\Rightarrow$ adj A is also symmetric.

A is diagonal $\Rightarrow$ adj A is also diagonal.

A is triangular $\Rightarrow$ adj A is also triangular.

A is singular $\Rightarrow |\text{adj } A| = 0$

INVERSE OF A SQUARE MATRIX

Let A be any n–rowed square matrix. Then a matrix B, if exists, such that $AB = BA = I_n$, is called the inverse of A. Inverse of A is usually denoted by A^{-1} (if exists).

We have, $|A|I_n = A(\text{adj}A)$

$|A|\, A^{-1} = (\text{adj } A)$. Thus the necessary and sufficient condition for a square matrix A to possess the inverse is that $|A| \neq 0$ and then $A^{-1} = \dfrac{\text{adj } (A)}{|A|}$.

A square matrix A is called non-singular if $|A| \neq 0$. Hence, a square matrix A is invertible if and only if A is non-singular.

Properties of Inverse of a Matrix

(i) $(A^{-1})^{-1} = A$

(ii) $(A^T)^{-1} = (A^{-1})^T$

(iii) $(AB)^{-1} = B^{-1}A^{-1}$

(iv) $(A^n)^{-1} = (A^{-1})^n,\ n \in \text{N}$

(v) $\text{adj } (A^{-1}) = (\text{adj } A)^{-1}$

$A = \text{diag}(a_1, a_2, ..., a_n)$. A is symmetric $\Rightarrow A^{-1}$ is also symmetric ($|A| \neq 0$).

A is diagonal matrix and $|A| \neq 0 \Rightarrow A^{-1}$ is also diagonal matrix.

A is scalar matrix $\Rightarrow A^{-1}$ is also a scalar matrix.

A is triangular matrix and $|A| \neq 0 \Rightarrow A^{-1}$ is also triangular matrix.

SYSTEM OF LINEAR SIMULTANEOUS EQUATIONS

Consider the system of linear non-homogeneous simultaneous equations in three unknowns x, y and z, given by $a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$ and $a_3x + b_3y + c_3z = d_3$

Let $A=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B=\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$,

Let $|A|=\Delta=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $\Delta_x=\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$,

obtained on replacing first column of Δ by B.

Similarly, let $\Delta_y=\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ and $\Delta_z=\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

It can be shown that $AX = B$, $x\Delta = \Delta_x$, $y\Delta = \Delta_y$, $z\Delta = \Delta_z$.

Determinant Method of Solution

We have the following two cases :

Case I

If $\Delta \neq 0$, then the given system of equations has unique solution, given by

$x = \Delta_x / \Delta$, $y = \Delta_y / \Delta$ and $z = \Delta_z / \Delta$.

Case II

If $\Delta = 0$, then two sub cases arise:

(a) at least one of Δ_x, Δ_y and Δ_z is non-zero, say $\Delta_x \neq 0$. Now in $x\cdot\Delta = \Delta_x$, L.H.S. is zero and R.H.S. is not equal to zero. Thus we have, no value of x satisfying $x\cdot\Delta = \Delta_x$.
Hence given system of equations has no solution.

(b) $\Delta_x = \Delta_y = \Delta_z = 0$. In the case the given equations are dependent. Delete one or two equation from the given system (as the case may be) to obtain independent equation(s). The remaining equation(s) may have no solution or infinitely many solution(s).

- **Matrix Method of Solution**

(a) $\Delta \neq 0$, then A^{-1} exists and hence
$AX = B \Rightarrow A^{-1}(AX) = A^{-1}B \Rightarrow X = A^{-1}B$ and therefore unique values of x, y and z are obtained.

(b) We have $AX = B \Rightarrow ((\text{adj } A)A)X = (\text{adj } A)B$
$\Rightarrow \Delta X = (\text{adj } A)B$.
If $\Delta = 0$, then $\Delta X = O_{3\times 1}$, zero matrix of order 3×1. Now if $(\text{adj } A)B = 0$, then the system $AX = B$ has infinitely many solutions, else no solution.

Note : A system of equation is called consistent if it has atleast one solution. If the system has no solution, then it is called inconsistent.

SYSTEM OF LINEAR HOMOGENEOUS SIMULTANEOUS EQUATIONS

Consider the system of linear homogeneous simultaneous equations in three unknowns x, y and z, given by $a_1x + b_1y + c_1z = 0$, $a_2x + b_2y + c_2z = 0$ and $a_3x + b_3y + c_3z = 0$.

In this case, system of equations is always consistent as $x = y = z = 0$ is always a solution. If the system has unique solution (the case when coefficient determinant $\neq 0$), then $x = y = z = 0$ is the only solution (called trivial solution). However if the system has coefficient determinant = 0, then the system has infinitely many solutions. Hence in this case we get solutions other than trivial solution also and we say that we have non-trivial solutions.

PROBLEMS

SECTION-I

Single Correct Answer Type

1. A and B are two non singular matrices so that $A^6 = I$ and $AB^2 = BA (B \neq I)$. A value of K so that $B^K = I$ is

(a) 31 (b) 32 (c) 63 (d) 64

2. For each real number x such that $-1 < x < 1$, let $A(x)$ be the matrix $(1-x)^{-1}\begin{bmatrix} 1 & -x \\ -x & 1 \end{bmatrix}$ and $z=\frac{x+y}{1+xy}$. Then,

(a) $A(z) = A(x) + A(y)$
(b) $A(z) = A(x)\,[A(y)]^{-1}$
(c) $A(z) = A(x)\,A(y)$
(d) $A(z) = A(x) - A(y)$

3. If $A=\begin{bmatrix} i & -i \\ -i & i \end{bmatrix}$ and $B=\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, then A^8 equals to

(a) $4B$ (b) $128B$ (c) $-128B$ (d) $-64B$

4. If $a - 2b + c = 1$, then the value of $\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix}$ is

(a) x (b) $-x$ (c) -1 (d) 1

5. If $A=\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$, $B=\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$, $C = ABA^T$, then A^TC^nA equals to $(n \in I^+)$

(a) $\begin{bmatrix} -n & 1 \\ 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -n \\ 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 \\ 1 & -n \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 \\ -n & 1 \end{bmatrix}$

6. If $p + q + r = 0$ and $\begin{vmatrix} pa & qb & rc \\ qc & ra & pb \\ rb & pc & qa \end{vmatrix} = k \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$, then $k =$

(a) 0 (b) abc
(c) pqr (d) $a + b + c$

7. $\begin{vmatrix} {}^{x}C_r & {}^{x}C_{r+1} & {}^{x}C_{r+2} \\ {}^{y}C_r & {}^{y}C_{r+1} & {}^{y}C_{r+2} \\ {}^{z}C_r & {}^{z}C_{r+1} & {}^{z}C_{r+2} \end{vmatrix} - \begin{vmatrix} {}^{x}C_r & {}^{x+1}C_{r+1} & {}^{x+2}C_{r+2} \\ {}^{y}C_r & {}^{y+1}C_{r+1} & {}^{y+2}C_{r+2} \\ {}^{z}C_r & {}^{z+1}C_{r+1} & {}^{z+2}C_{r+2} \end{vmatrix} =$

(a) 0 (b) 2^n
(c) ${}^{x+y+z}C_r$ (d) ${}^{x+y+z}C_{r+2}$

8. If $a, b, c, d > 0$; $x \in R$ and $(a^2 + b^2 + c^2)x^2 - 2(ab + bc + cd)\, x + b^2 + c^2 + d^2 \le 0$, then $\begin{vmatrix} 33 & 14 & \log a \\ 65 & 27 & \log b \\ 97 & 40 & \log c \end{vmatrix} =$

(a) 1 (b) – 1
(c) 0 (d) none of these

9. If A and B are two non singular matrices and both are symmetric and commute each other then

(a) Both $A^{-1}B$ and $A^{-1}B^{-1}$ are symmetric.
(b) $A^{-1}B$ is symmetric but $A^{-1}B^{-1}$ is not symmetric.
(c) $A^{-1}B^{-1}$ is symmetric but $A^{-1}B$ is not symmetric.
(d) Neither $A^{-1}B$ nor $A^{-1}B^{-1}$ are symmetric.

10. If $f(x) = ax^2 + bx + c$, $a, b, c \in R$ and the equation $f(x) - x = 0$ has imaginary roots α and β and γ and δ be the roots of $f(f(x)) - x = 0$, then $\begin{vmatrix} 2 & \alpha & \delta \\ \beta & 0 & \alpha \\ \gamma & \beta & 1 \end{vmatrix}$ is

(a) 0 (b) purely real
(c) purely imaginary (d) none of these

11. Suppose a matrix A satisfies $A^2 - 5A + 7I = 0$. If $A^5 = aA + bI$, then the value of $2a + b$ is

(a) –87 (b) –105
(c) 1453 (d) 1155

12. If 'α' is a root of $x^4 = 1$ with negative principal argument, then the principal argument of $\Delta(\alpha)$ where $\Delta(\alpha) = \begin{vmatrix} 1 & 1 & 1 \\ \alpha^n & \alpha^{n+1} & \alpha^{n+3} \\ \frac{1}{\alpha^{n+1}} & \frac{1}{\alpha^n} & 0 \end{vmatrix}$ is

(a) $\frac{5\pi}{14}$ (b) $-\frac{3\pi}{4}$ (c) $\frac{\pi}{4}$ (d) $-\frac{\pi}{4}$

13. If z is a complex number and $l_1, l_2, l_3, m_1, m_2, m_3$ are all real, then $\begin{vmatrix} l_1 z + m_1 \bar{z} & m_1 z + l_1 \bar{z} & m_1 z + l_1 \\ l_2 z + m_2 \bar{z} & m_2 z + l_2 \bar{z} & m_2 z + l_2 \\ l_3 z + m_3 \bar{z} & m_3 z + l_3 \bar{z} & m_3 z + l_3 \end{vmatrix}$ is equal to

(a) $|z|^2$
(b) 3
(c) $(l_1 l_2 l_3 + m_1 m_2 m_3)^2 |z|^2$
(d) 0

14. Let $x > 0, y > 0, z > 0$ are respectively the 2nd, 3rd, 4th terms of a G.P and $\Delta = \begin{vmatrix} x^k & x^{k+1} & x^{k+2} \\ y^k & y^{k+1} & y^{k+2} \\ z^k & z^{k+1} & z^{k+2} \end{vmatrix} = (r-1)^2 \left(1 - \frac{1}{r^2}\right)$ (where r is the common ratio) then

(a) $k = -1$ (b) $k = 1$
(c) $k = 0$ (d) None of these

15. Let a, b, c be positive real numbers. Then the following system of equations in x, y, z $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, -\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ has

(a) no solution (b) unique solution
(c) infinite solution (d) finitely many solution

16. If $\begin{vmatrix} a^2 + \lambda^2 & ab + c\lambda & ca - b\lambda \\ ab - c\lambda & b^2 + \lambda^2 & bc + a\lambda \\ ac + b\lambda & bc - a\lambda & c^2 + \lambda^2 \end{vmatrix} \begin{vmatrix} \lambda & c & -b \\ -c & \lambda & a \\ b & -a & \lambda \end{vmatrix} = (1 + a^2 + b^2 + c^2)^3$ then λ is equal to

(a) 0 (b) 1 (c) –1 (d) ±2

17. The system of equations $x + ky + 3z = 0$, $3x + ky - 2z = 0$, $2x + 3y - 4z = 0$ possess a non-trivial solution over the set of rationals, then $2k$ is an integral element of the interval

(a) [10, 20] (b) (20, 30)
(c) [30, 40] (d) (40, 50)

18. If A and B are two square matrices of order 3×3 which satisfy $AB = A$ and $BA = B$ then $(A + B)^7$ is

(a) $7(A + B)$ (b) $7I_{3\times3}$
(c) $64(A + B)$ (d) $128\, I_{3\times3}$

19. A and B are square matrices and A is non-singular matrix, $(A^{-1}BA)^n$, $n \in I^+$, is equal to

(a) $A^{-n}B^nA^n$ (b) $A^nB^nA^{-n}$
(c) $A^{-1}B^nA$ (d) $A^{-n}BA^n$

SECTION-II

Multiple Correct Answer Type

20. If $\Delta = \begin{vmatrix} a & c & c-a & a+c \\ c & b & b-c & b+c \\ a-b & b-c & 0 & a-c \\ x & y & z & 1+x+y \end{vmatrix} = 0$, then

(a) a, b, c are in A.P. (b) b, a, c are in A.P.
(c) a, c, b are in G.P. (d) a, c, b are in H.P.

21. If 'A' is a matrix of size $n \times n$ such that $A^2 + A + 2I = 0$, then

(a) A is non-singular (b) A is symmetric
(c) $|A| \neq 0$ (d) $A^{-1} = \frac{-1}{2}(A + I)$

22. Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, then

(a) $A^2 - 4A - 5I_3 = 0$ (b) $A^{-1} = \frac{1}{5}(A - 4I_3)$
(c) A^3 is not invertible (d) A^2 is invertible

23. If the sum of two idempotent matrices is idempotent then

(a) $AB + BA = 0$ (b) $AB = BA = I$
(c) $AB \neq BA$ (d) $AB - BA = 0$

24. If a square matrix $A = [a_{ij}]$, $a_{ij} = i^2 - j^2$ is of even order, then

(a) A is a skew symmetric matrix
(b) $|A|$ is a perfect square
(c) A is a symmetric and $|A| = 0$
(d) A is neither symmetric nor skew symmetric

25. If

$$\Delta(x) = \begin{vmatrix} x^2 + 4x - 3 & 2x + 4 & 13 \\ 2x^2 + 5x - 9 & 4x + 5 & 26 \\ 8x^2 - 6x + 1 & 16x - 6 & 104 \end{vmatrix} = ax^3 + bx^2 + cx + d,$$

then

(a) $a = 3$ (b) $a = 0$
(c) $c = 0$ (d) $b = 2$

26. If $f(x) = \begin{vmatrix} (1+x)^a & (1+2x)^b & 1 \\ 1 & (1+x)^a & (1+2x)^b \\ (1+2x)^b & 1 & (1+x)^a \end{vmatrix}$, where a, b being positive integers, then

(a) constant term of $f(x)$ is 0
(b) coefficient of x in $f(x)$ is 0
(c) constant term in $f(x)$ is $(a - b)$
(d) coefficient of x in $f(x)$ is $(a - b)$

27. The value of the determinant

$$\begin{vmatrix} \cos(\theta + \alpha) & -\sin(\theta + \alpha) & \cos 2\alpha \\ \sin\theta & \cos\theta & \sin\alpha \\ -\cos\theta & \sin\theta & \lambda\cos\alpha \end{vmatrix}$$ is

(a) independent of θ for all $\lambda \in R$
(b) independent of θ and α when $\lambda = 1$
(c) independent of θ and α when $\lambda = -1$
(d) independent of λ for all θ

28. Suppose that a, b, c are real numbers such that $a + b + c = 1$. If the matrix $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$ be an orthogonal matrix, then

(a) A is an involutory matrix
(b) $|A|$ is negative
(c) $a^3 + b^3 + c^3 - 3abc = 1$
(d) atleast one of a, b, c is negative

29. If A is a square matrix such that

$A \cdot (\text{adj}A) = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ then

(a) $|A| = 4$ (b) $|\text{adj}A| = 16$
(c) $\frac{|\text{adj}(\text{adj}A)|}{|\text{adj}A|} = 16$ (d) $|\text{adj } 2A| = 128$

30. System of equations $x + 3y + 2z = 6$, $x + \lambda y + 2z = 7$ and $x + 3y + 2z = \mu$ has

(a) unique solution if $\lambda = 2, \mu \neq 6$
(b) infinitely many solution, if $\lambda = 4, \mu = 6$
(c) no solution if $\lambda = 5, \mu = 7$
(d) no solution if $\lambda = 3, \mu = 5$

31. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, then

(a) $A^3 = 9A$ (b) $A^3 = 27A$
(c) $A + A = A^2$ (d) A^{-1} does not exist

SECTION-III

Comprehension Type

Paragraph for Question No. 32 to 34

A Pythagorean triplet is triplet of positive integers (a, b, c) such that $a^2 + b^2 = c^2$. Define the matrices P, Q and R by

$P = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & -2 \\ 2 & 2 & 3 \end{bmatrix}$ and $R = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix}$

32. If we write Pythagorean triplets (a, b, c) in matrix form as $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ then which of the following matrix product is a Pythagorean triplet ?

(a) $Q\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ (b) $P\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ (c) $R\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ (d) none of these

33. Which one of the following does not hold good ?

(a) $P^{-1} = \text{adj}.P$ (b) $(PQ)^{-1} = \text{adj}.(PQ)$

(c) $(QR)^{-1} = \text{adj}.(QR)$ (d) $(PQR)^{-1} \neq \text{adj}.(PQR)$

34. $T_r(P + Q^T + 2R)$ equals

(a) 17 (b) 15

(c) 14 (d) 18

Paragraph for Question No. 35 to 37

If A is 3×3 matrix then a non trivial solution $X = (x \;\; y \;\; z)^T$ such that $AX = \lambda X(\lambda \in R)$ yields 3 values of λ say $\lambda_1, \lambda_2, \lambda_3$. For any such matrix A, λ's are called eigen values and corresponding X's are called eigen vectors. It is known that, for any 3×3 matrix $\text{Tr}(A) = \lambda_1 + \lambda_2 + \lambda_3$, $\det A = \lambda_1\lambda_2\lambda_3$. Answer the following questions for matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

35. $\text{Tr}(A^{-1}) =$

(a) $\frac{1}{3}$ (b) $\frac{1}{2}$ (c) $-\frac{1}{2}$ (d) $-\frac{1}{3}$

36. $\text{Tr}(A^3) =$

(a) 149 (b) 101

(c) 128 (d) 133

37. Which of the following is false?

(a) $\exists$ a non trivial solution X such that $AX = (2+\sqrt{7})X$

(b) $\exists$ a non trivial solution X such that $AX = X$

(c) $\exists$ a non trivial solution X such that $A^{-1}X = (2-\sqrt{7})X$

(d) The total number of non trivial solutions X such that $AX = \lambda X$ is 3.

Paragraph for Question No. 38 to 40

Let $\Delta \neq 0$ and Δ^c denotes the determinant of cofactors, then $\Delta^c = \Delta^{n-1}$, where n (> 0) is the order of Δ.

On the basis of above information, answer the following questions:

38. If a, b, c are the roots of the equation $x^3 - px^2 + r = 0$, then the value of

$$\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}$$ is

(a) p^2 (b) p^4 (c) p^6 (d) p^9

39. If a, b, c are the roots of the equation $x^3 - 3x^2 + 3x + 7 = 0$, then the value of

$$\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ac-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}$$ is

(a) 9 (b) 27 (c) 81 (d) 0

40. Suppose $a, b, c \in R$, $a + b + c > 0$, $A = bc - a^2$, $B = ca - b^2$ and $C = ab - c^2$ and $\begin{vmatrix} A & B & C \\ B & C & A \\ C & A & B \end{vmatrix} = 49$, then $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ equals

(a) -7 (b) 7

(c) -2401 (d) 2401

SECTION-IV

Matrix-Match Type

41. Let $p(\theta) = \begin{vmatrix} -\sqrt{2} & \sin\theta & \cos\theta \\ 1 & \cos\theta & \sin\theta \\ -1 & \sin\theta & -\cos\theta \end{vmatrix}$,

$q(\theta) = \begin{vmatrix} \sin 2\theta & -1 & 1 \\ \cos 2\theta & 4 & -3 \\ 2 & 7 & -5 \end{vmatrix}$; $r(\theta) = \begin{vmatrix} \cos\theta & \sin\theta & \cos\theta \\ -\sin\theta & \cos\theta & \sin\theta \\ -\cos\theta & -\sin\theta & \cos\theta \end{vmatrix}$

and $s(\theta) = \begin{vmatrix} \sec^2\theta & 1 & 1 \\ \cos^2\theta & \cos^2\theta & \text{cosec}^2\theta \\ 1 & \cos^2\theta & \cot^2\theta \end{vmatrix}$

Match the functions on the left with their range on the right.

Column I		Column II	
(A)	$p(\theta)$	(p)	$[0, 1]$
(B)	$q(\theta)$	(q)	$[0, 2\sqrt{2}]$
(C)	$r(\theta)$	(r)	$[-2, 2]$
(D)	$s(\theta)$	(s)	$[-\sqrt{5}-2, \sqrt{5}-2]$

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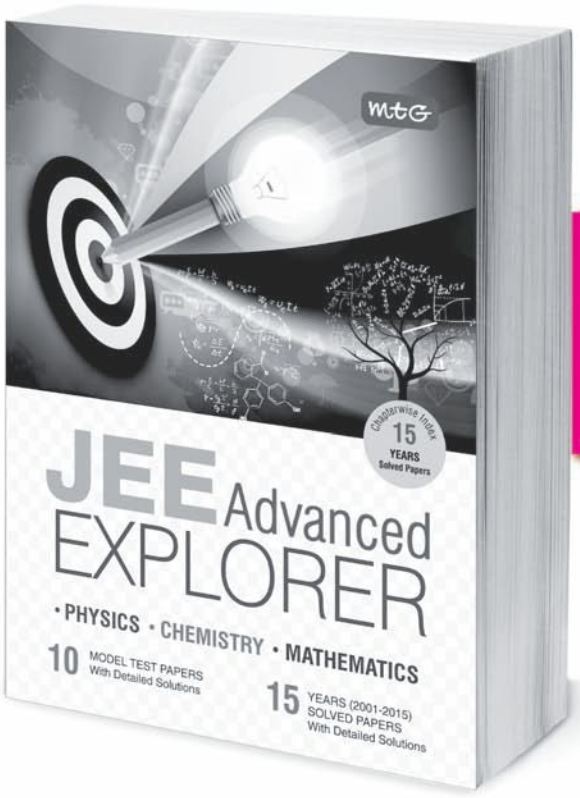

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42.

Column I		Column II	
(A)	A is a matrix such that $A^2 = A$. If $(I + A)^8 = I + \lambda A$, then $\lambda + 1$ is equal to	(p)	64
(B)	If A is a square matrix of order 3 such that $\|A\| = 2$, then $(\text{adj}A^{-1})^{-1}$ is equal to	(q)	1
(C)	Let $\|A\| = \|a_{ij}\|_{3\times 3} \neq 0$. Each element a_{ij} is multiplied by λ^{i-j}. Let $\|B\|$ is the resulting determinant, where $\|A\| = \lambda \|B\|$, then λ is equal to	(r)	256
(D)	If A is a diagonal matrix of order 3×3 is commutative with every square matrix of order 3×3 under multiplication and trace $(A) = 12$, then $\|A\| =$	(s)	4

43. A, B are two matrices of 3×3 order such that

Column I		Column II	
(A)	A is non singular and $AB = 0$ then	(p)	$A = 0$
(B)	B is non singular and $AB = 0$ then	(q)	$B = 0$
(C)	A & B both are non zero matrix and $AB = 0$ then	(r)	det $A = 0$
(D)	$A^n = 0$ for some $n \geq 2$ then	(s)	det $B = 0$

SECTION-V

Integer Answer Type

44. Let A_n, $(n \in N)$ be a matrix of order $(2n - 1) \times (2n - 1)$, such that $a_{ij} = 0 \ \forall \ i \neq j$ and $a_{ij} = n^2 + i + 1 - 2n \ \forall \ i = j$ where a_{ij} denotes the element of i^{th} row and j^{th} column of A_n.
Let $T_n = (-1)^n \times$ (sum of all the elements of A_n). Find the value of $\left[\dfrac{\sum_{n=1}^{102} T_n}{520200}\right]$, where [.] represents the greatest integer function.

45. Find the value of $f\left(\dfrac{\pi}{6}\right)$, where

$$f(\theta) = \begin{vmatrix} \cos^2\theta & \cos\theta\sin\theta & -\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta & \cos\theta \\ \sin\theta & -\cos\theta & 0 \end{vmatrix}$$

46. $\det P = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$, where '$P$' is an orthogonal matrix.
Then the value of $|a + b + c|$ is

47. If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then $\det(A^{2005})$ equals to

48. If x, y, z are cube roots of unity and

$$D = \begin{vmatrix} x^2 + y^2 & z^2 & z^2 \\ x^2 & y^2 + z^2 & x^2 \\ y^2 & y^2 & z^2 + x^2 \end{vmatrix},$$

then the real part of D is

49. Find the coefficient of x in the determinant

$$\begin{vmatrix} (1+x)^{a_1b_1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3} \\ (1+x)^{a_2b_1} & (1+x)^{a_2b_2} & (1+x)^{a_2b_3} \\ (1+x)^{a_3b_1} & (1+x)^{a_3b_2} & (1+x)^{a_3b_3} \end{vmatrix},$$

where $a_i, b_i \in N$ $(i = 1, 2, 3)$

50. If $f(x) = \begin{vmatrix} \cos(x+\alpha) & \cos(x+\beta) & \cos(x+\gamma) \\ \sin(x+\alpha) & \sin(x+\beta) & \sin(x+\gamma) \\ \sin(\beta-\gamma) & \sin(\gamma-\alpha) & \sin(\alpha-\beta) \end{vmatrix}$

and $f(2) = 6$, then find $\sum_{r=1}^{25} f(r)/30$.

51. If $f(x)$ satisfies the equation

$$\begin{vmatrix} f(x+1) & f(x+8) & f(x+1) \\ 1 & 2 & -5 \\ 2 & 3 & \lambda \end{vmatrix} = 0$$

for all real x. If f is periodic with period 7, then find the value of $|\lambda|$.

52. If $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -3 \\ 2 & 1 & 0 \end{bmatrix}$, $B = (\text{adj } A)$ and $C = 5A$, then find the value of $\dfrac{|\text{adj } B|}{|C|}$.

53. If $A = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$ and $A^2 = 8A + kI_2$, then find the value of $|k|$.

SOLUTIONS

1. **(c) :** $A^5(AB^2) = A^5BA$
$\Rightarrow B^2 = A^5BA$
$\Rightarrow B^4 = (A^5BA)(A^5BA) = A^5B^2A = A^5(A^5BA)A$
$\Rightarrow B^4 = A^4BA^2$

$\Rightarrow B^8 = (A^4BA^2)(A^4BA^2) = A^4B^2A^2 = A^4(A^5BA)A^2$
$\Rightarrow B^8 = A^3BA^3$
$\Rightarrow B^{16} = (A^3BA^3)(A^3BA^3) = A^3B^2A^3 = A^3(A^5BA)A^3 = A^2BA^4$
$\Rightarrow B^{32} = (A^2BA^4)(A^2BA^4) = A^2B^2A^4 = A^2(A^5BA)A^4 = ABA^5$
$\Rightarrow B^{64} = (ABA^5)(ABA^5) = AB^2A^5 = A(A^5BA)A^5 = B$
$\Rightarrow B^{63} = I$

2. (c) : $A(z) = A\left(\frac{x+y}{1+xy}\right)$

$$= \left[\frac{1+xy}{(1-x)(1-y)}\right]\begin{bmatrix} 1 & -\left(\frac{x+y}{1+xy}\right) \\ -\left(\frac{x+y}{1+xy}\right) & 1 \end{bmatrix}$$

$\therefore A(x)\,A(y) = A(z)$

3. (b) : We have, $A = iB$

$\Rightarrow A^2 = (iB)^2 = i^2B^2 = -B^2 = -\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = -2B$

$\Rightarrow A^4 = (-2B)^2 = 4B^2 = 4(2B) = 8B$

$\Rightarrow (A^4)^2 = (8B)^2 \Rightarrow A^8 = 64B^2 = 128B$

4. (c) : $\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix}$ since $a-2b+c=1$ or $(a-b)+(c-b)=1$

Apply the operation,
$R_1 \to R_1 - 2R_2 + R_3$
$R_3 \to R_3 - R_2$, the determinant reduces to

$$\begin{vmatrix} 0 & 0 & 1 \\ x+2 & x+3 & x+b \\ 1 & 1 & c-b \end{vmatrix} = -1$$

5. (d) : $A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$

$AA^T = I$ ... (i)
Now, $C = ABA^T \Rightarrow A^TC = BA^T$ ... (ii)
Now, $A^TC^nA = A^TC\cdot C^{n-1}A = BA^TC^{n-1}A$ (from (ii))
$= BA^TC\,C^{n-2}A = B^2A^TC^{n-2}A = ...$
$= B^{n-1}A^T\,CA = B^{n-1}BA^TA = B^n = \begin{bmatrix} 1 & 0 \\ -n & 1 \end{bmatrix}$

6. (c) : $p+q+r=0 \Rightarrow p^3+q^3+r^3 = 3pqr$

$$\begin{vmatrix} pa & qb & rc \\ qc & ra & pb \\ rb & pc & qa \end{vmatrix} = pqr(a^3+b^3+c^3-3abc)$$

$$= pqr\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} \Rightarrow k = pqr$$

7. (a) : $\begin{vmatrix} {}^xC_r & {}^xC_{r+1} & {}^xC_{r+2} \\ {}^yC_r & {}^yC_{r+1} & {}^yC_{r+2} \\ {}^zC_r & {}^zC_{r+1} & {}^zC_{r+2} \end{vmatrix}$

$$= \begin{vmatrix} {}^xC_r & {}^{x+1}C_{r+1} & {}^{x+1}C_{r+2} \\ {}^yC_r & {}^{y+1}C_{r+1} & {}^{y+1}C_{r+2} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z+1}C_{r+2} \end{vmatrix}$$

By applying $C_2 \to C_2 + C_1$, $C_3 \to C_3 + C_2$
Now apply $C_3 \to C_3 - C_2$, we get

$$\begin{vmatrix} {}^xC_r & {}^{x+1}C_{r+1} & {}^{x+2}C_{r+2} \\ {}^yC_r & {}^{y+1}C_{r+1} & {}^{y+2}C_{r+2} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z+2}C_{r+2} \end{vmatrix} = 0$$

8. (c) : We have
$(a^2+b^2+c^2)x^2 - 2(ab+bc+cd)x + b^2+c^2+d^2 \le 0$
$\Rightarrow (ax-b)^2 + (bx-c)^2 + (cx-d)^2 \le 0$
$\Rightarrow (ax-b)^2 + (bx-c)^2 + (cx-d)^2 = 0$
$\Rightarrow \frac{b}{a} = \frac{c}{b} = \frac{d}{c} = x$
$\Rightarrow b^2 = ac$ or $2\log b = \log a + \log c$,

Now, $\begin{vmatrix} 33 & 14 & \log a \\ 65 & 27 & \log b \\ 97 & 40 & \log c \end{vmatrix} = \begin{vmatrix} 130 & 54 & \log a + \log c \\ 65 & 27 & \log b \\ 97 & 40 & \log c \end{vmatrix}$ [Apply $R_1 \to R_1 + R_3$]

$\Rightarrow \begin{vmatrix} 0 & 0 & 0 \\ 65 & 27 & \log b \\ 97 & 40 & \log c \end{vmatrix} = 0$ [Apply $R_1 \to R_1 - 2R_2$]

9. (a) : $AB = BA$
Pre & post multiplying both sides by A^{-1}.
$A^{-1}(AB)A^{-1} = A^{-1}(BA)A^{-1}$
$(A^{-1}A)(BA^{-1}) = A^{-1}B(AA^{-1})$
$\Rightarrow (BA^{-1})' = (A^{-1}B)' = (A^{-1})'B'$ (reversal law)
$= A^{-1}B$ (as $B = B'$ and $(A^{-1})' = A^{-1}$)
$\Rightarrow A^{-1}B$ is symmetric.
Similarly for $A^{-1}B^{-1}$.

10. (b) : $f(x) - x > 0$ or, $f(x) - x < 0 \;\forall\; x \in R$
$f(f(x)) - f(x) > 0$ or, $f(f(x)) - f(x) < 0$
Adding, $f(f(x)) - x > 0$ or, $f(f(x)) - x < 0$
$\Rightarrow$ Roots of $f(f(x)) - x = 0$ are imaginary.

Let $z = \begin{vmatrix} 2 & \alpha & \delta \\ \beta & 0 & \alpha \\ \gamma & \beta & 1 \end{vmatrix} \Rightarrow \bar{z} = \begin{vmatrix} 2 & \bar{\alpha} & \bar{\delta} \\ \bar{\beta} & 0 & \bar{\alpha} \\ \bar{\gamma} & \bar{\beta} & 1 \end{vmatrix} = \begin{vmatrix} 2 & \beta & \gamma \\ \alpha & 0 & \beta \\ \delta & \alpha & 1 \end{vmatrix} = z$

11. (a) : $A^3 = AA^2 = A(5A - 7I)$
$= 5A^2 - 7A = 5(5A - 7I) - 7A = 18A - 35I$

$A^4 = A \cdot A^3 = A(18A - 35I) = 18(5A - 7I) - 35A$
$= 55A - 126I$
$A^5 = A \cdot A^4 = A(55A - 126I) = 55(5A - 7I) - 126A$
$= 149A - 385I$
$\Rightarrow a = 149, b = -385$

12. (b) : Clearly $\alpha = -i$, where $i^2 = -1$

So $\Delta(\alpha) = \alpha^n \frac{1}{\alpha^n}\begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^3 \\ \frac{1}{\alpha} & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -i & i \\ i & 1 & 0 \end{vmatrix}$

$= 1(-i) + 1(i^2) + (1 + i^2) = -1 - i$

So, principal argument of $\Delta(\alpha)$ is $-\frac{3\pi}{4}$.

13. (d) : $\begin{vmatrix} l_1 & m_1 & 0 \\ l_2 & m_2 & 0 \\ l_3 & m_3 & 0 \end{vmatrix} \times \begin{vmatrix} z & \bar{z} & 1 \\ \bar{z} & z & z \\ 1 & z & 0 \end{vmatrix} = 0$

14. (a) : $x^k y^k z^k \begin{vmatrix} 1 & ar & a^2r^2 \\ 1 & ar^2 & a^2r^4 \\ 1 & ar^3 & a^2r^6 \end{vmatrix}$

$= a^{3(k+1)} \cdot r^{3(2k+1)}[(r - 1)(r^4 - 1) - (r^2 - 1)^2]$
$\Rightarrow k = -1$

15. (d) : Let $\frac{x^2}{a^2} = X, \frac{y^2}{b^2} = Y, \frac{z^2}{c^2} = Z$

$X + Y - Z = 1, X - Y + Z = 1, -X + Y + Z = 1$
On solving $X = Y = Z = 1$
$\Rightarrow x = \pm a, y = \pm b, z = \pm c \Rightarrow$ 8 solutions

16. (b) : If $\Delta = \begin{vmatrix} \lambda & c & -b \\ -c & \lambda & a \\ b & -a & \lambda \end{vmatrix}$, other determinant (say Δ') is the cofactor determinant.
$\Delta\Delta' = \Delta^3$ (for 3rd order det.)
$\Delta = \lambda(\lambda^2 + a^2 + b^2 + c^2)$, by comparing $\lambda = 1$

17. (c) : For the given system to have a non-trivial solution, we must have

$\begin{vmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 3 & -4 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 1 & k & 3 \\ 0 & -2k & -11 \\ 0 & 3-2k & -10 \end{vmatrix} = 0$

[Applying $R_2 \to R_2 - 3R_1, R_3 \to R_3 - 2R_1$]

$\Rightarrow 20k + 11(3 - 2k) = 0 \Rightarrow k = \frac{33}{2}$

18. (c) : $AB = A, BA = B \Rightarrow A^2 = A$ and $B^2 = B$
$(A + B)^2 = A^2 + B^2 + AB + BA$
$= A + B + A + B = 2(A + B)$
$(A + B)^3 = (A + B)^2(A + B) = 2(A + B)^2 = 2^2(A + B)$
$\therefore (A + B)^7 = 2^6(A + B) = 64(A + B)$

19. (c) : For $n = 2 \Rightarrow (A^{-1}BA)^2 = (A^{-1}BA)(A^{-1}BA)$
$= A^{-1}B^2A$
Also $(A^{-1}BA)^3 = (A^{-1}B^3A)$ and so on
Thus $(A^{-1}BA)^n = A^{-1}B^nA$

20. (a, c) : $C_3 \to C_3 - (C_2 - C_1)$
$C_4 \to C_4 - (C_1 + C_2)$

$\begin{vmatrix} a & c & 0 & 0 \\ c & b & 0 & 0 \\ a-b & b-c & a+c-2b & 0 \\ x & y & z+x-y & 1 \end{vmatrix} = (a+c-2b)(ab-c^2)$

21. (a, c, d) : $A(A + I) = -2I$
$|A(A + I)| = |-2I|$
$|A|\,|A + I| = (-2)^n \neq 0$

22. (a, b, d) : $A^2 - 4A - 5I_3$

$= \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} + \begin{bmatrix} -4 & -8 & -8 \\ -8 & -4 & -8 \\ -8 & -8 & -4 \end{bmatrix} + \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$

$\therefore A^2 - 4A - 5I_3 = 0$
or $A^{-1}A^2 - 4A^{-1}A - 5A^{-1}I_3 = 0$
or $(A^{-1}A)A - 4I_3 - 5A^{-1} = 0$
or $IA - 4I_3 - 5A^{-1} = 0$

$\therefore A^{-1} = \frac{1}{5}(A - 4I_3)$

Also, $|A^2| = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$

$= 9(81 - 64) - 8(72 - 64) + 8(64 - 72)$
$= (9 \times 17) - (8 \times 8) + 8 \times (-8)$
$= 25 \neq 0$
$\therefore A^2$ is invertible.
and $A^3 = A \cdot A^2 = A(4A + 5I_3) = 4A^2 + 5A$

$= \begin{bmatrix} 36 & 32 & 32 \\ 32 & 36 & 32 \\ 32 & 32 & 36 \end{bmatrix} + \begin{bmatrix} 5 & 10 & 10 \\ 10 & 5 & 10 \\ 10 & 10 & 5 \end{bmatrix} = \begin{bmatrix} 41 & 42 & 42 \\ 42 & 41 & 42 \\ 42 & 42 & 41 \end{bmatrix}$

$\therefore |A^3| \neq 0$
$\therefore A^3$ is invertible.

23. (a, d) : $(A + B)^2 = A^2 + AB + BA + B^2 = A + B$

$\Rightarrow AB + BA = 0$

$A^2B + ABA = 0 \Rightarrow A^2B + (-BA)A = 0$

$\Rightarrow AB - BA = 0$

24. (a, b) : Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}$

$$A' = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix} = -A$$

A is skew-symmetric matrix and $|A| = \begin{vmatrix} 0 & -3 \\ 3 & 0 \end{vmatrix} = 9$ is a perfect square.

25. (b, c) : $\Delta'(x) = \begin{vmatrix} 2x+4 & 2x+4 & 13 \\ 4x+5 & 4x+5 & 26 \\ 16x-6 & 16x-6 & 104 \end{vmatrix}$

$$+ \begin{vmatrix} x^2+4x-3 & 2 & 13 \\ 2x^2+5x-9 & 4 & 26 \\ 8x^2-6x+1 & 16 & 104 \end{vmatrix}$$

$= 0 + 2 \times 13 \times (0) = 0$

$\Rightarrow \Delta(x) =$ constant $\Rightarrow a = 0, b = 0, c = 0$

26. (a, b) : Let

$$\begin{vmatrix} (1+x)^a & (1+2x)^b & 1 \\ 1 & (1+x)^a & (1+2x)^b \\ (1+2x)^b & 1 & (1+x)^a \end{vmatrix} = A + Bx + Cx^2 + \ldots$$

Putting $x = 0$, we get $A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$

Now differentiating both sides w.r.t. x and putting $x = 0$, we get

$$B = \begin{vmatrix} a & 2b & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 0 & a & 2b \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2b & 0 & a \end{vmatrix} = 0$$

Hence coefficient of x is 0.

27. (a, c) : We have,

$$\begin{vmatrix} \cos(\theta+\alpha) & -\sin(\theta+\alpha) & \cos 2\alpha \\ \sin\theta & \cos\theta & \sin\alpha \\ -\cos\theta & \sin\theta & \lambda\cos\alpha \end{vmatrix}$$

$$= \frac{1}{\sin\alpha\cos\alpha} \begin{vmatrix} \cos(\theta+\alpha) & -\sin(\theta+\alpha) & \cos 2\alpha \\ \sin\theta\sin\alpha & \cos\theta\sin\alpha & \sin^2\alpha \\ -\cos\theta\cos\alpha & \sin\theta\cos\alpha & \lambda\cos^2\alpha \end{vmatrix}$$

[Multiplying R_2 and R_3 by sinα and cosα, respectively]

$$= \frac{1}{\sin\alpha\cos\alpha} \begin{vmatrix} 0 & 0 & \cos 2\alpha + \sin^2\alpha + \lambda\cos^2\alpha \\ \sin\theta\sin\alpha & \cos\theta\sin\alpha & \sin^2\alpha \\ -\cos\theta\cos\alpha & \sin\theta\cos\alpha & \lambda\cos^2\alpha \end{vmatrix}$$

[Applying $R_1 \to R_1 + R_2 + R_3$]

$$= \frac{\cos 2\alpha + \sin^2\alpha + \lambda\cos^2\alpha}{\sin\alpha\cdot\cos\alpha} \begin{vmatrix} \sin\theta\sin\alpha & \cos\theta\sin\alpha \\ -\cos\theta\cos\alpha & \sin\theta\cos\alpha \end{vmatrix}$$

$$= (\cos^2\alpha + \lambda\cos^2\alpha) \begin{vmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{vmatrix}$$

$= (1 + \lambda)\cos^2\alpha$

Therefore, the given determinant is independent of θ for all real values of λ. Also, $\lambda = -1$, then it is independent of θ and α.

28. (a, b, c, d) : $AA^T = A^TA = I$.

Also $A^T = A$, so $A^2 = I \Rightarrow A$ is involutory matrix.

$\Rightarrow |A^2| = |A|^2 = 1$ or, $|A| = \pm 1$.

But $|A| = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = (a+b+c) \begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$

$= (a + b + c)(ab + bc + ca - a^2 - b^2 - c^2)$

$|A| = ab + bc + ca - a^2 - b^2 - c^2 \quad (\because a + b + c = 1)$

$\therefore \quad a^2 + b^2 + c^2 - ab - bc - ca \geq 0$

So $|A| = -1$. Hence $a^3 + b^3 + c^3 - 3abc = 1$.

Again $a^2 + b^2 + c^2 - ab - bc - ca = 1$

$\Rightarrow 1 - 3(ab + bc + ca) = 1$, so $ab + bc + ca = 0$,

$\Rightarrow$ atleast one of a, b, and c is negative.

29. (a, b, c) : $A(\text{adj}A) = |A|I \therefore |A| = 4$

$|\text{adj}A| = |A|^{n-1} = 16$;

$|\text{adj}(\text{adj}A)| = |A|^{(n-1)^2} = |A|^4 = 256$

$\text{adj}(KA) = K^{n-1}(\text{adj}A)$

$\therefore \quad |\text{adj}(KA)| = (K^{n-1})^n |\text{adj}A|$

$\therefore \quad |\text{adj}(2A)| = 2^6 \cdot 16$

30. (b, c, d) : $x + 3y + 2z = 6 \quad \ldots(i)$

$x + \lambda y + 2z = 7 \quad \ldots(ii)$

$x + 3y + 2z = \mu \quad \ldots(iii)$

(a) If $\lambda = 2$, then $D = 0$, therefore unique solution is not possible.

(b) If $\lambda = 4$, $\mu = 6$

$x + 3y = 6 - 2z$

$x + 4y = 7 - 2z$

$y = 1$ and $x = 3 - 2z$

Substituting in equation (iii)

$3 - 2z + 3 + 2z = 6$ is satisfied

$\therefore$ Infinite solutions.

(c) $\lambda = 5$, $\mu = 7$

Consider equation (ii) and (iii)

$x + 5y = 7 - 2z$

$x + 3y = 7 - 2z$

$y = 0$, $x = 7 - 2z$ are solutions.

Substitute in (i)

$7 - 2z + 2z = 6$, which does not satisfy.

$\therefore$ No solution.

(d) If $\lambda = 3$, $\mu = 5$

then equation (i) and (ii) have no solution.

$\therefore$ No solution .

31. (a, d) : $A^2 = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix} = 3A$

$A^3 = A^2A = 3A \cdot A = 3A^2 = 3(3A) = 9A$ and $|A| = 0$

$\therefore$ A^{-1} does not exist.

32. (d) : $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 26 \\ 20 \\ 29 \end{bmatrix} \Rightarrow 29^2 \neq 20^2 + 26^2$

Similarly, $P\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ and $Q\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$

do not form pythagorean triplet

33. (d) : $\det P = 1$, $\det Q = 1$, $\det R = 1$

$\det(PQ) = 1$, $\det(QR) = 1$, $\det(RP) = 1$

$\det(PQR) = 1$

34. (c) : $\text{Tr}(P + Q^T + 2R) = 14$

35. (d) **36. (b)** **37. (c)**

For the matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$,

$|A - \lambda I| = 0 \Rightarrow \lambda^3 - 5\lambda^2 + \lambda + 3 = 0$ ... (1)

whose roots are $\lambda_1, \lambda_2, \lambda_3$

$\lambda_1 + \lambda_2 + \lambda_3 = 5$, $\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = 1$, $\lambda_1\lambda_2\lambda_3 = -3$

$\Rightarrow$ $\text{Tr}(A) = 5$, $\det(A) = -3$

Also, $AX = \lambda X \Rightarrow A^{-1}X = \frac{1}{\lambda}X$

$\Rightarrow \text{Tr}(A^{-1}) = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = \frac{\sum \lambda_1\lambda_2}{\lambda_1\lambda_2\lambda_3} = -\frac{1}{3}$

$AX = \lambda X \Rightarrow A^3X = \lambda^3 X \Rightarrow \text{Tr}(A^3) = \lambda_1^3 + \lambda_2^3 + \lambda_3^3$

$= (\lambda_1 + \lambda_2 + \lambda_3)(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - \lambda_1\lambda_2 - \lambda_2\lambda_3 - \lambda_3\lambda_1) + 3\lambda_1\lambda_2\lambda_3$

$= 5(25 - 3(1)) - 9 = 101$

Solving (1) gives $\lambda_1 = 1$, $\lambda_2 = 2 + \sqrt{7}$, $\lambda_3 = 2 - \sqrt{7}$, which by theory yield non trivial solutions. In particular for A^{-1}, the values of λ yieldings non trivial solutions are

$1, \frac{1}{2+\sqrt{7}}, \frac{1}{2-\sqrt{7}}$ *i.e.* $1, \frac{2+\sqrt{7}}{-3}, \frac{2-\sqrt{7}}{-3}$.

Hence (c) is false.

38. (c) : $a + b + c = p$, $ab + bc + ca = 0$

$\therefore$ $a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca)$

$= p^2 - 0 = p^2$

If $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$\therefore$ $\Delta^c = \begin{vmatrix} bc - a^2 & ca - b^2 & ab - c^2 \\ ca - b^2 & ab - c^2 & bc - a^2 \\ ab - c^2 & bc - a^2 & ca - b^2 \end{vmatrix} = \Delta^{3-1}$

Now, $\Delta^2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$= \begin{vmatrix} a^2 + b^2 + c^2 & ab + bc + ca & ab + bc + ca \\ ab + bc + ca & a^2 + b^2 + c^2 & ab + bc + ca \\ ab + bc + ca & ab + bc + ca & a^2 + b^2 + c^2 \end{vmatrix}$

$= \begin{vmatrix} p^2 & 0 & 0 \\ 0 & p^2 & 0 \\ 0 & 0 & p^2 \end{vmatrix} = p^6$

39. (d) : $\because$ $x^3 - 3x^2 + 3x + 7 = 0$

$\Rightarrow (x - 1)^3 + 8 = 0 \Rightarrow (x - 1)^3 = (-2)^3$

$\Rightarrow \left(\frac{x-1}{-2}\right)^3 = 1 \Rightarrow \frac{x-1}{-2} = (1)^{1/3} = 1, \omega, \omega^2$

$\Rightarrow x - 1 = -2, -2\omega, -2\omega^2$

or $x = -1, 1 - 2\omega, 1 - 2\omega^2$

$\therefore$ $a = -1$, $b = 1 - 2\omega$, $c = 1 - 2\omega^2$

$\because$ $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$= \begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix} \times \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ (row by row)

$= \begin{vmatrix} 2bc - a^2 & c^2 & b^2 \\ c^2 & 2ac - b^2 & a^2 \\ b^2 & a^2 & 2ab - c^2 \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$

$= (a^3 + b^3 + c^3 - 3abc)^2$

$= \{(a+b+c)(a^2+b^2+c^2-ab-bc-ca)\}^2$

$= \frac{1}{4}(a+b+c)^2\left\{(a-b)^2+(b-c)^2+(c-a)^2\right\}^2$

$= \frac{9}{4}\{8(\omega-1)^2(1+\omega+\omega^2)\} = 0$

40. (a) : $\begin{vmatrix} A & B & C \\ B & C & A \\ C & A & B \end{vmatrix} = \begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}$

$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = 49 \quad \therefore \quad \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -7$

41. A → q; B → s; C → r; D → p

(A) Expand along C_1 to obtain

$p(\theta) = (-\sqrt{2})(-1) + (-1)(-2\sin\theta\cos\theta) + (-1)(\sin^2\theta - \cos^2\theta)$

$= \sqrt{2} + \sin 2\theta + \cos 2\theta = \sqrt{2} + \sqrt{2}\sin\left(2\theta + \frac{\pi}{4}\right)$

$\therefore$ Range of $p(\theta)$ is $\left[0, 2\sqrt{2}\right]$

(B) Applying $R_2 \to R_2 + 4R_1$, $R_3 \to R_3 + 7R_1$, we get

$q(\theta) = \begin{vmatrix} \sin 2\theta & -1 & 1 \\ \cos 2\theta + 4\sin 2\theta & 0 & 1 \\ 2 + 7\sin 2\theta & 0 & 2 \end{vmatrix}$

$= 2\cos 2\theta + 8\sin 2\theta - 2 - 7\sin 2\theta$

$= 2\cos 2\theta + \sin 2\theta - 2$

As $2\cos 2\theta + \sin 2\theta$ lies between $-\sqrt{5}$ to $\sqrt{5}$, we get range of $q(\theta)$ is $[-\sqrt{5}-2, \sqrt{5}-2]$

(C) Using $C_1 \to C_1 + C_3$, we get

$r(\theta) = 2\cos\theta \begin{vmatrix} 1 & \sin\theta & \cos\theta \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{vmatrix} = 2\cos\theta$

$\therefore$ Range of $r(\theta)$ is $[-2, 2]$

(D) Taking $\sec^2\theta$ common from R_1, we get

$s(\theta) = \sec^2\theta \begin{vmatrix} 1 & \cos^2\theta & \cos^2\theta \\ \cos^2\theta & \cos^2\theta & \operatorname{cosec}^2\theta \\ 1 & \cos^2\theta & \cot^2\theta \end{vmatrix}$

$R_3 \to R_3 - R_1$, we get

$s(\theta) = \sec^2\theta \begin{vmatrix} 1 & \cos^2\theta & \cos^2\theta \\ \cos^2\theta & \cos^2\theta & \operatorname{cosec}^2\theta \\ 0 & 0 & \cot^2\theta - \cos^2\theta \end{vmatrix}$

$= \sec^2\theta\,(\cot^2\theta - \cos^2\theta)(\cos^2\theta - \cos^4\theta)$

$= (\cot^2\theta - \cos^2\theta)\sin^2\theta = \cos^2\theta - \cos^2\theta\sin^2\theta = \cos^4\theta$

$\therefore$ Range of $s(\theta)$ is $[0, 1]$

42. A → r; B → s; C → q; D → p

(A) $(I+A)^8 = {}^8C_0 I + {}^8C_1 IA + {}^8C_2 IA^2 + \ldots\ldots + {}^8C_8 IA^8$

$= {}^8C_0 I + {}^8C_1 A + {}^8C_2 A + \ldots\ldots + {}^8C_8 A$

$= I + A({}^8C_1 + {}^8C_2 + \ldots\ldots + {}^8C_8)$

$= I + A(2^8 - 1) \Rightarrow \lambda = 2^8 - 1$

(B) $\left|\text{adj}(A^{-1})\right| = |A^{-1}|^2 = \frac{1}{|A|^2}$

$\left|(\text{adj}(A^{-1}))^{-1}\right| = \frac{1}{|\text{adj } A^{-1}|} = |A|^2 = 2^2 = 4$

(C) $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$\Rightarrow \quad |B| = \begin{vmatrix} a_{11} & \lambda^{-1}a_{12} & \lambda^{-2}a_{13} \\ \lambda a_{21} & a_{22} & \lambda^{-1}a_{23} \\ \lambda^2 a_{31} & \lambda a_{32} & a_{33} \end{vmatrix}$

$= \frac{1}{\lambda^3}\begin{vmatrix} \lambda^2 a_{11} & \lambda a_{12} & a_{13} \\ \lambda^2 a_{21} & \lambda a_{22} & a_{23} \\ \lambda^2 a_{31} & \lambda a_{32} & a_{33} \end{vmatrix} = |A|$

Hence, $|A| = |B| \Rightarrow \lambda = 1$.

(D) A diagonal matrix is commutative with every square matrix, if it is a scalar matrix.

So, every diagonal element is 4.

$\therefore \quad |A| = 64$.

43. A → q, s; B → p, r; C → r, s; D → r

(A) $\because |A| \neq 0 \Rightarrow A^{-1}$ exist.

$\therefore AB = 0 \Rightarrow A^{-1}AB = 0 \Rightarrow IB = 0 \Rightarrow B = 0$

(B) $AB = 0$, $|B| \neq 0 \Rightarrow B^{-1}$ exist.

$ABB^{-1} = 0 \Rightarrow AI = 0 \Rightarrow A = 0$

(C) $AB = 0 \Rightarrow |A|\,|B| = 0 \Rightarrow$ three cases

(i) either $|A| = 0, |B| \neq 0$

or (ii) $|A| \neq 0, |B| = 0$ or (iii) $|A| = 0, |B| = 0$

(i) $|B| \neq 0, AB = 0 \Rightarrow A = 0$ but in question $A \neq 0$

$\Rightarrow$ Ist case is not possible.

(ii) case is also not possible, hence both $|A|, |B|$ should be zero.

(D) $A^n = 0 \Rightarrow |A^n| = 0 \Rightarrow |A| = 0$

44. (2) : $a_{ij} = 0 \ \forall\ i \neq j$ and $a_{ij} = (n-1)^2 + i \ \ \forall\ i = j$

Sum of all the elements of $A_n = \sum_{i=1}^{2n-1}[(n-1)^2 + i]$

$= (2n-1)(n-1)^2 + (2n-1)n = 2n^3 - 3n^2 + 3n - 1$

$= n^3 + (n-1)^3$

So, $T_n = (-1)^n [n^3 + (n-1)^3]$

$= (-1)^n n^3 - (-1)^{n-1}(n-1)^3 = V_n - V_{n-1}$

$$\Rightarrow \quad \sum_{n=1}^{102} T_n = \sum_{n=1}^{102} (V_n - V_{n-1}) = V_{102} - V_0 = (102)^3$$

$$\left[\frac{\sum_{n=1}^{102} T_n}{520200}\right] = 2$$

45. (1) : Applying $C_1 \to C_1 - \sin\theta\, C_3$ and $C_2 \to C_2 + \cos\theta\, C_3$, we get

$$f(\theta) = \begin{vmatrix} 1 & 0 & -\sin\theta \\ 0 & 1 & \cos\theta \\ \sin\theta & -\cos\theta & 0 \end{vmatrix}$$

Applying $R_3 \to R_3 - \sin\theta\, R_1 + \cos\theta\, R_2$, we get

$$f(\theta) = \begin{vmatrix} 1 & 0 & -\sin\theta \\ 0 & 1 & \cos\theta \\ 0 & 0 & \sin^2\theta + \cos^2\theta \end{vmatrix} = 1$$

Thus, $f\left(\frac{\pi}{6}\right) = 1$

46. (1) : $PP^T = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}\begin{bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} a^2+b^2+c^2 & ab+bc+ca & ab+bc+ca \\ ab+bc+ca & a^2+b^2+c^2 & ab+bc+ca \\ ab+bc+ca & ab+bc+ca & a^2+b^2+c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

47. (1) : $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -8 \\ 2 & -3 \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} 5 & -8 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 7 & -12 \\ 3 & -5 \end{bmatrix}$$

Observing A, A^2, A^3 we can conclude that

$$A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$$

$$\det(A^n) = \begin{vmatrix} 1+2n & -4n \\ n & 1-2n \end{vmatrix} = 1 - 4n^2 + 4n^2 = 1$$

$\det(A^{2005}) = 1$

48. (4) : Applying $R_1 \to R_1 - R_2 - R_3$, we get

$$D = \begin{vmatrix} 0 & -2y^2 & -2x^2 \\ x^2 & y^2+z^2 & x^2 \\ y^2 & y^2 & z^2+x^2 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$= -2\begin{vmatrix} 0 & y^2 & x^2 \\ x^2 & z^2 & 0 \\ y^2 & 0 & z^2 \end{vmatrix}$$

$= 4x^2y^2z^2 = 4\,(1\cdot\omega\cdot\omega^2)^2 \;= 4$

49. (0) : Let $\begin{vmatrix} (1+x)^{a_1b_1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3} \\ (1+x)^{a_2b_1} & (1+x)^{a_2b_2} & (1+x)^{a_2b_3} \\ (1+x)^{a_3b_1} & (1+x)^{a_3b_2} & (1+x)^{a_3b_3} \end{vmatrix}$

$$= \lambda_0 + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 + \ldots$$

For λ_1, differentiate w.r.t. x and put $x = 0$

So $\lambda_1 = 0$

50. (5) : Clearly $f'(x) = 0$

$\therefore \quad f(x) = c = 6$

$$\therefore \quad \sum_{r=1}^{25} f(r) = \sum_{r=1}^{25} 6 = 150$$

51. (4) : On solving, we get

$(2\lambda + 15) f(x+1) - (\lambda + 10) f(x+8) - f(x+1) = 0$

$(2\lambda + 14) f(x+1) = (\lambda + 10) f(x+8)$

Since f is periodic with period 7

$\therefore \quad f(x+1) = f(x+8)$

$\Rightarrow \quad 2\lambda + 14 = \lambda + 10 \quad \Rightarrow \quad |\lambda| = 4$

52. (1) : $\frac{|\text{adj } B|}{|C|} = \frac{|\text{adj}(\text{adj } A)|}{|5A|} = \frac{|A|^{(3-1)^2}}{5^3|A|} = \frac{|A|^3}{125}$

Now $|A| = 5$

$\therefore \quad \frac{|\text{adj } B|}{|C|} = 1$

53. (7) : Here $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 \\ -1 & 7-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad (1-\lambda)(7-\lambda) = 0$

$\Rightarrow \quad \lambda^2 - 8\lambda + 7 = 0$

$\Rightarrow \quad A^2 - 8A + 7I_2 = 0$

$\Rightarrow \quad A^2 = 8A - 7I_2$

$\Rightarrow \quad k = -7 \quad \Rightarrow \quad |k| = 7$

ACE YOUR WAY CBSE XII

Series 3

MATRICES AND DETERMINANTS

INTRODUCTION

A set of mn numbers arranged in a rectangular array of m horizontal lines and n vertical lines, enclosed by [ ] or ( ), is called a matrix of order m by n, written as $m \times n$ matrix.

The numbers which form a matrix are called entries or elements of the matrix. The horizontal lines in a matrix are called rows and the vertical lines are called columns of the matrix.

The element in the i^{th} row and j^{th} column of a matrix A is denoted by a_{ij}. The first suffix indicates the row and the second suffix indicates the column to which that element belongs. With this notation a matrix A of order $m \times n$ can be written as:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

or, $A = [a_{ij}]_{m \times n}$; $i = 1, 2, ..., m$; $j = 1, 2, ..., n$ or simply $A = [a_{ij}]_{m \times n}$

EQUALITY OF MATRICES

Two matrices $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ are said to be equal if

(i) they have the same order, *i.e.*, the number of rows and columns in both the matrices are same.

(ii) their corresponding elements are equal, *i.e.*, $a_{ij} = b_{ij}$ for each i and j.

Thus, two matrices are equal if and only if one is a duplicate of the other.

TYPES OF MATRICES

1. **Row matrix :** A matrix having exactly one row and any number of columns is called a row matrix.
2. **Column matrix :** A matrix having exactly one column and any number of rows is called a column matrix.
3. **Square matrix :** A matrix having same number of rows and columns is called a square matrix.
4. **Diagonal matrix :** A square matrix is called a diagonal matrix if all its non-diagonal elements are zero.

   i.e., the square matrix $[a_{ij}]_{m \times n}$ is a diagonal matrix if $a_{ij} = 0$ for $i \neq j$.
5. **Scalar matrix :** A square matrix is a called a scalar matrix if its non-diagonal elements are zero and diagonal elements are equal.

   i.e., the square matrix $[a_{ij}]_{m \times n}$ is a scalar matrix if

   $$a_{ij} = \begin{cases} k, & \text{for } i = j \\ 0, & \text{for } i \neq j \end{cases}$$
6. **Unit matrix or identity matrix :** A square matrix is called a unit matrix if its non-diagonal elements are zero and diagonal elements are all equal to one (unity).

   i.e., the square matrix $[a_{ij}]_{m \times n}$ is a unit matrix if

   $$a_{ij} = \begin{cases} 1, & \text{for } i = j \\ 0, & \text{for } i \neq j \end{cases}$$
7. **Null (or zero) matrix :** A matrix having all the elements zero is called a null matrix or zero matrix. The zero matrix of order $m \times n$ is denoted by the symbol $O_{m \times n}$.
8. **Upper-triangular matrix :** A square matrix is called an upper-triangular matrix if all the elements below the principal diagonal are zero.

   i.e., the square matrix $[a_{ij}]_{m \times n}$ is an upper-triangular matrix if $a_{ij} = 0$ for all $i > j$.
9. **Lower-triangular matrix :** A square matrix is called a lower-triangular matrix if all the elements above the principal diagonal are zero.

   i.e., the square matrix $[a_{ij}]_{m \times n}$ is a lower-triangular matrix if $a_{ij} = 0$ for all $i < j$.

10. **Triangular matrix :** A square matrix is called a triangular matrix if it is either an upper-triangular matrix or a lower-triangular matrix.
i.e., the square matrix $[a_{ij}]_{m\times n}$ is a triangular matrix if $a_{ij} = 0$ for all $i > j$ (or all $i < j$).
11. **Comparable matrix :** Two matrix A and B are said to be comparable if they are of the same order, *i.e.*, they have the same number of rows and the same number of columns.

ALGEBRA OF MATRICES

1. **Addition of matrices :** Let A and B be two comparable matrices, each of order $(m \times n)$. Then, their sum $(A + B)$ is a matrix of order $(m \times n)$, obtained by adding the corresponding elements of A and B.
i.e., if $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ then $A + B = [a_{ij} + b_{ij}]_{m\times n}$.
2. **Negative of a matrix :** Let $A = [a_{ij}]_{m\times n}$. Then, the negative of A is the matrix $(-A) = [-a_{ij}]_{m\times n}$, obtained by replacing each element of A with its corresponding additive inverse. $(-A)$ is called the additive inverse of A.
3. **Subtraction of matrices :** If A and B are two comparable matrices, then we define $(A - B) = A + (-B)$.
4. **Scalar multiplication :** Let A be a given matrix and k be a number. Then, the matrix obtained by multiplying each element of A by k is called the scalar multiple of A by k, to be denoted by kA.
If A is an $(m \times n)$ matrix, then kA is also an $(m \times n)$ matrix.
If $A = [a_{ij}]_{m\times n}$, then $kA = [k \cdot a_{ij}]_{m\times n}$.
5. **Multiplication of matrices :** Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{jk}]_{n\times p}$ be two matrices, then the product $AB = C = [c_{ik}]_{m\times p}$ is called multiplication of two matrices where $c_{ik} = a_{i1}\, b_{1k} + a_{i2}\, b_{2k} + \ldots + a_{in}\, b_{nk}$.

PROPERTIES OF MATRIX ADDITION AND MULTIPLICATION

S.No.	Properties of matrix Addition and Multiplication	
1.	Commutative	If A and B are two matrices then $A + B = B + A$ and $AB \neq BA$
2.	Associative	If A, B and C are three matrices, then $(A + B) + C = A + (B + C)$ and $(AB)C = A(BC)$
3.	Distributive	If A, B & C are three matrices, then $A(B + C) = AB + AC$ and $(A + B)C = AC + BC$
4.	Existence of identity	The null matrix is the identity element for matrix addition *i.e.*, $A + O = A = O + A$. If A is a square matrix, then identity element for matrix multiplication is I, *i.e.*, $IA = A = AI$, of same order.
5.	Existence of inverse	For every matrix $A = [a_{ij}]_{m\times n}$, there exists a matrix $[-a_{ij}]_{m\times n}$ denoted by $-A$, such that $A + (-A) = O = (-A) + A$

MATRIX POLYNOMIAL

Let $f(x) = a_0x^m + a_1x^{m-1} + \ldots + a_{m-1}x + a_m$ be a polynomial in x and A be a square matrix of order n, then $f(A) = a_0A^m + a_1A^{m-1} + \ldots + a_{m-1}A + a_mI_n$ is called a matrix polynomial in A. Thus, to obtain $f(A)$, replace x by A in $f(x)$ and the constant term is multiplied by the identity matrix of the order equal to that of A.
The polynomial equation $f(x) = 0$ is said to be satisfied by the matrix A iff $f(A) = O$.

TRANSPOSE OF A MATRIX

Let $A = [a_{ij}]$ be an $m \times n$ matrix. Then, the transpose of A, denoted by A^T or A', is an $n \times m$ matrix such that $A^T = [a_{ji}]$ for all $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$.

Thus, A^T is obtained from A by changing its rows into columns and its columns into rows.
For example, if

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 2 & 1 & 4 \end{bmatrix}, \text{ then } A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 4 & 1 \\ 4 & 1 & 4 \end{bmatrix}$$

Properties of transpose
For any two matrices A and B of the same order,
(i) $(A^T)^T = A$
(ii) $(A + B)^T = A^T + B^T$
(iii) $(kA)^T = k(A^T)$, where k is a scalar.
(iv) $(AB)^T = B^TA^T$

SOME SPECIAL MATRICES

- **Symmetric matrix :** A square matrix $A = [a_{ij}]_{m\times n}$ is called a symmetric matrix, if $A = A^T$ *i.e.*, $a_{ij} = a_{ji}$ for all i, j
- **Skew-symmetric matrix :** A square matrix $A = [a_{ij}]_{m\times n}$ is a skew-symmetric matrix if $A^T = -A$ *i.e.*, $a_{ij} = -a_{ji}$ for all i, j
- **Orthogonal matrix :** A square matrix A of $m \times n$ order is called an orthogonal matrix if $AA^T = A^TA = I$.

ELEMENTARY OPERATION (TRANSFORMATION) OF A MATRIX

The following three operations applied on the rows (columns) of a matrix are called elementary row (column) operations:

(i) **Interchange of any two rows (columns) :** If i^{th} row (column) is interchanged with j^{th} row (column), we write $R_i \leftrightarrow R_j$ ($C_i \leftrightarrow C_j$).

(ii) **Multiplying the elements of a row (column) by a non-zero scalar :** If the elements of i^{th} row (column) are multiplied by a non-zero scalar k, we write $R_i \to kR_i$ ($C_i \to kC_i$).

(iii) **Adding to the elements of a row (column), the constant times the corresponding elements of another row (column) :** If k times the elements of j^{th} row (column) are added to the corresponding elements of the i^{th} row (column), we write $R_i \to R_i + kR_j$ ($C_i \to C_i + kC_j$).

Invertible matrices

Let A be a square matrix of order n. Then a square matrix B of order n, if it exists, is called an inverse of A if $AB = BA = I$. In that case A is said to be invertible. The inverse of an invertible matrix A is denoted by A^{-1}.

Uniqueness of inverse

Inverse of a square matrix, if it exists, is unique.

Inverse of a matrix by elementary operations

Let A is a matrix of order n. To find A^{-1} using elementary row operations, write $A = IA$ and apply a sequence of row operations on $A = IA$ till we get $I = BA$. The matrix B will be the inverse of A. Similarly, if we wish to find A^{-1} using column operations, then write $A = AI$ and apply a sequence of column operations on $A = AI$ till we get $I = AB$.

DETERMINANTS

A determinant is a pure number associated with a square matrix, denoted by det A or $|A|$, written as

$$|A| = \det A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

A determinant having n rows and n columns is called a determinant of order n.

Determinant of a square matrix of order 1

Let $A = [a_{11}]$ be a 1×1 matrix, then the determinant of A is the number a_{11} itself, *i.e.*, $|a_{11}| = a_{11}$.

Determinant of a square matrix of order 2

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a 2×2 matrix, then

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

i.e. the determinant of a 2×2 matrix is obtained by taking the product of the entries on the main diagonal and subtracting from it the product of the entries in the other diagonal.

Notes :

- $\det I_n = 1$, where I_n is unit/identity matrix of order n.
- $\det O_n = 0$, O is null matrix (square matrix of n order).
- $A = [a_{ij}]_{n \times n}$, then $|A| = |A'|$ {Reflection Property}.
- $\det (AB) = \det A \cdot \det B$, where A & B are matrices of same order.
- $\det (kA) = k^n \det A$, if A is of order $n \times n$.
- $\det (A^n) = (\det A)^n$ if $n \in I^+$.

MINORS AND COFACTORS

- The minor of an element a_{ij} in $|A|$ is defined as the value of the determinant obtained by deleting the i^{th} row and j^{th} column of $|A|$, and it is denoted by M_{ij}.
- The cofactor C_{ij} of an element a_{ij} is defined as $C_{ij} = (-1)^{i+j} \cdot M_{ij}$.

PROPERTIES OF DETERMINANTS

- The value of a determinant remains unaltered if the rows and columns are interchanged.
- If two rows (or columns) of a determinant are interchanged, the value of the determinant is multiplied by –1.
- If any two rows (or columns) of a determinant are identical, then the value of determinant is zero.
- If the elements of a row (or column) of a determinant are multiplied by a scalar, then the value of the new determinant is equal to same scalar times the value of the original determinant.
- If each element of any row (or column) of a determinant is the sum of two numbers, then the determinant is expressible as the sum of two determinants of the same order.

Evaluation of determinants using elementary operations

We shall use the following notations to evaluate determinant

1. The operation of interchanging the i^{th} row and j^{th} row will be denoted by $R_i \to R_j$.
2. The operation of multiplying each element of the i^{th} row by a number k will be denoted by $R_i \to kR_i$.
3. The operation of adding to each element of the i^{th} row, k times the corresponding element of the j^{th} row ($j \neq i$) will be denoted by $R_i \to R_i + kR_j$.

Similarly notations are used for operations on columns replacing R by C.

APPLICATION OF DETERMINANTS

- Area of triangle with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ is given by $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$.
- If area is zero, then three points are collinear.

ADJOINT OF A MATRIX

Let $A = [a_{ij}]$ be a square matrix of order n and let C_{ij} denote the cofactors of a_{ij} in $|A|$. Then, the adjoint of A, denoted by adj A, is defined as adj $A = [C_{ji}]_{n \times n}$.

Singular and non-singular matrices

A square matrix A is said to be
(i) singular if $|A| = 0$, (ii) non-singular if $|A| \neq 0$.

PROPERTIES OF ADJOINT OF A MATRIX

- For every square matrix of order n
  $A(\text{adj } A) = (\text{adj } A)A = |A| \cdot I$
- If $|A| = 0$, matrix A is called singular else non singular.
- If A and B are non singular matrices of the same order, then AB and BA are also non singular matrices of same order.
- (adj AB) = (adj B)·(adj A)
- (adj A)′ = adj A'
- If A be $n \times n$ matrix, then
  (i) $|\text{adj } A| = |A|^{n-1}$
  (ii) $\text{adj}(\text{adj } A) = |A|^{n-2}A$
  (iii) $|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$

INVERSE OF A SQUARE MATRIX

If A be any n-rowed square matrix, then a matrix B, if it exists such that $AB = I_n = BA$, is called inverse of A. The inverse of A is denoted by A^{-1}.

$$A^{-1} = \frac{1}{|A|}(\text{adj } A)$$

Properties of the inverse of a matrix

1. A square matrix is invertible if and only if it is non-singular.
2. The inverse of the inverse is the original matrix itself, *i.e.*, $(A^{-1})^{-1} = A$.
3. The inverse of the transpose of a matrix is the transpose of its inverse, *i.e.* $(A')^{-1} = (A^{-1})'$.
4. (Reversal law) If A and B are invertible matrices of the same order, then AB is also invertible and $(AB)^{-1} = B^{-1}A^{-1}$.
5. If A, B, C are invertible matrices of same order, then $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

SOLUTION OF SYSTEM OF EQUATIONS

- Let $AX = B$ be the given system of equations :
  (i) If $|A| \neq 0$, the system is consistent and has one unique solution.
  (ii) If $|A| = 0$ and (adj A)$B \neq O$, then the system is inconsistent so it has no solution.
  (iii) If $|A| = 0$ and (adj A)$B = O$, then the system may be either consistent or inconsistent according as the system name either infinitely many solutions or no solution.

Very Short Answer Type

1. If a matrix has 12 elements, what are the possible orders it can have?

2. Simplify :

$$\cos\theta\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta\begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$$

3. Show that $\begin{vmatrix} b-c & c-a & a-b \\ c-a & a-b & b-c \\ a-b & b-c & c-a \end{vmatrix} = 0$

4. Find matrix X, if $X + \begin{bmatrix} 4 & 6 \\ -3 & 7 \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ 5 & -8 \end{bmatrix}$.

5. If $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$, find adj. A.

Short Answer Type

6. Using properties of determinants, prove that

$$\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix} = x^3$$

7. Prove that : $\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix} = 1$

8. Using elementary transformations, find the inverse of $\begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$.

9. Verify that $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ satisfies the equation $A^3 - 4A^2 + A = 0$.

10. If $f(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$,
show that $[f(x)]^{-1} = f(-x)$.

Long Answer Type

11. A trust fund has ₹ 30,000 that must be invested in two different types of bonds. The first bond pays 5% interest per year, and the second bond pays 7% interest per year. Using matrix multiplication, determine how to divide ₹ 30,000 among the two types of bonds if the trust fund must obtain an annual total interest of (i) ₹ 1800 (ii) ₹ 2000.

12. Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$. Prove that $A^2 - 4A - 5I = O$.

and also find A^{-1}.

13. If $x \neq y \neq z$ and $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$, then prove that $xyz = -1$.

14. Find the matrix P satisfying the matrix equation

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} P \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

15. Discuss the consistency of the following system of equations:

$$x + y + z = 1$$
$$2x + 2y + 2z = 2$$
$$3x + 3y + 3z = 4.$$

SOLUTIONS

1. If a matrix has mn elements, then its order is $m \times n$ or $n \times m$.
Here, $mn = 12 = (1)\ (12)$ or $(2)\ (6)$ or $(3)\ (4)$.
Possible orders of the matrix are : $1 \times 12, 12 \times 1, 2 \times 6, 6 \times 2, 3 \times 4, 4 \times 3$.

2. We have,

$$\cos\theta \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ -\sin\theta\cos\theta & \cos^2\theta \end{bmatrix} + \begin{bmatrix} \sin^2\theta & -\sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \sin\theta\cos\theta - \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & \cos^2\theta + \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3. Let $\Delta = \begin{vmatrix} b-c & c-a & a-b \\ c-a & a-b & b-c \\ a-b & b-c & c-a \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} 0 & c-a & a-b \\ 0 & a-b & b-c \\ 0 & b-c & c-a \end{vmatrix} = 0 \ [\because\ C_1 \text{ consists of all zeros}]$$

4. Let $A = \begin{bmatrix} 4 & 6 \\ -3 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -6 \\ 5 & -8 \end{bmatrix}$.

Then, the given matrix equation is $X + A = B$
$\Rightarrow X = B + (-A)$

$$= \begin{bmatrix} 3 & -6 \\ 5 & -8 \end{bmatrix} + \begin{bmatrix} -4 & -6 \\ 3 & -7 \end{bmatrix}.$$

$$= \begin{bmatrix} 3+(-4) & -6+(-6) \\ 5+3 & -8+(-7) \end{bmatrix} = \begin{bmatrix} -1 & -12 \\ 8 & -15 \end{bmatrix}$$

Hence, $X = \begin{bmatrix} -1 & -12 \\ 8 & -15 \end{bmatrix}$

5. The cofactors of the elements of $|A|$ are given by $A_{11} = 3$, $A_{12} = -1$, $A_{21} = -5$, $A_{22} = 2$.

$$\therefore \quad \text{adj } A = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}^T = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

6. Let the given determinant be Δ. Then,

$$\Delta = \begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix}$$

Applying $R_2 \to R_2 - 2R_1$ & $R_3 \to R_3 - 3R_1$

$$\Delta = \begin{vmatrix} x+y & x & x \\ 3x+2y & 2x & 0 \\ 7x+5y & 5x & 0 \end{vmatrix}$$

Expanding along C_3, we get

$$\Delta = x \cdot \begin{vmatrix} 3x+2y & 2x \\ 7x+5y & 5x \end{vmatrix}$$

Taking x common from C_2

$$\Delta = x^2 \cdot \begin{vmatrix} 3x+2y & 2 \\ 7x+5y & 5 \end{vmatrix}$$

$= x^2 \cdot [5(3x + 2y) - 2(7x + 5y)] = (x^2 \cdot x) = x^3$
Hence, $\Delta = x^3$

7. Let the given determinant be Δ. Then,

$$\Delta = \begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$$

Multiplying R_1 by $\sin\alpha$, R_3 by $\cos\alpha$ and dividing Δ by $\sin\alpha\cos\alpha$

$$\Delta = \frac{1}{\sin\alpha\cos\alpha}\cdot\begin{vmatrix}\sin\alpha\cos\alpha\cos\beta & \sin\alpha\cos\alpha\sin\beta & -\sin^2\alpha\\ -\sin\beta & \cos\beta & 0\\ \sin\alpha\cos\alpha\cos\beta & \sin\alpha\cos\alpha\sin\beta & \cos^2\alpha\end{vmatrix}$$

Applying $R_1 \to R_1 - R_3$

$$\Delta = \frac{1}{\sin\alpha\cos\alpha}\cdot\begin{vmatrix}0 & 0 & -1\\ -\sin\beta & \cos\beta & 0\\ \sin\alpha\cos\alpha\cos\beta & \sin\alpha\cos\alpha\sin\beta & \cos^2\alpha\end{vmatrix}$$

Expanding along R_1

$$\Delta = \frac{-1}{\sin\alpha\cos\alpha}\cdot[-\sin\alpha\cos\alpha\sin^2\beta - \sin\alpha\cos\alpha\cos^2\beta]$$

$$= \frac{-1}{\sin\alpha\cos\alpha}\cdot(-\sin\alpha\cos\alpha)[\sin^2\beta + \cos^2\beta] = 1$$

Hence Proved.

8. In order to use elementary row operations we write

$A = IA$, where $A = \begin{bmatrix}2 & -6\\ 1 & -2\end{bmatrix}$ and $I = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

We have,

$$\begin{bmatrix}2 & -6\\ 1 & -2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}A$$

$$\Rightarrow \begin{bmatrix}1 & -4\\ 1 & -2\end{bmatrix} = \begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}A, \text{ applying } R_1 \to R_1 - R_2$$

$$\Rightarrow \begin{bmatrix}1 & -4\\ 0 & 2\end{bmatrix} = \begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix}A, \text{ applying } R_2 \to R_2 - R_1$$

$$\Rightarrow \begin{bmatrix}1 & -4\\ 0 & 1\end{bmatrix} = \begin{bmatrix}1 & -1\\ -\frac{1}{2} & 1\end{bmatrix}A, \text{ applying } R_2 \to \frac{1}{2}R_2$$

$$\Rightarrow \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}-1 & 3\\ -\frac{1}{2} & 1\end{bmatrix}A, \text{ applying } R_1 \to R_1 + 4R_2$$

Thus, $A^{-1} = \begin{bmatrix}-1 & 3\\ -\frac{1}{2} & 1\end{bmatrix}$

9. We have, $A = \begin{bmatrix}2 & 3\\ 1 & 2\end{bmatrix}$

$$A^2 = \begin{bmatrix}2 & 3\\ 1 & 2\end{bmatrix}\begin{bmatrix}2 & 3\\ 1 & 2\end{bmatrix} = \begin{bmatrix}4+3 & 6+6\\ 2+2 & 3+4\end{bmatrix} = \begin{bmatrix}7 & 12\\ 4 & 7\end{bmatrix}$$

$$A^3 = A^2A = \begin{bmatrix}7 & 12\\ 4 & 7\end{bmatrix}\begin{bmatrix}2 & 3\\ 1 & 2\end{bmatrix}$$

$$= \begin{bmatrix}14+12 & 21+24\\ 8+7 & 12+14\end{bmatrix} = \begin{bmatrix}26 & 45\\ 15 & 26\end{bmatrix}$$

$$\therefore\ A^3 - 4A^2 + A = \begin{bmatrix}26 & 45\\ 15 & 26\end{bmatrix} - 4\begin{bmatrix}7 & 12\\ 4 & 7\end{bmatrix} + \begin{bmatrix}2 & 3\\ 1 & 2\end{bmatrix}$$

$$= \begin{bmatrix}26 & 45\\ 15 & 26\end{bmatrix} - \begin{bmatrix}28 & 48\\ 16 & 28\end{bmatrix} + \begin{bmatrix}2 & 3\\ 1 & 2\end{bmatrix}$$

$$= \begin{bmatrix}26-28+2 & 45-48+3\\ 15-16+1 & 26-28+2\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

Hence, $A^3 - 4A^2 + A = 0$

10. We have, $f(x) = \begin{bmatrix}\cos x & -\sin x & 0\\ \sin x & \cos x & 0\\ 0 & 0 & 1\end{bmatrix}$

$$\Rightarrow\ f(-x) = \begin{bmatrix}\cos(-x) & -\sin(-x) & 0\\ \sin(-x) & \cos(-x) & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}\cos x & \sin x & 0\\ -\sin x & \cos x & 0\\ 0 & 0 & 1\end{bmatrix} \quad \text{...(i)}$$

Let $f(x) = A$, then

$$|A| = \begin{bmatrix}\cos x & -\sin x & 0\\ \sin x & \cos x & 0\\ 0 & 0 & 1\end{bmatrix}$$

$= (\cos^2 x + \sin^2 x) = 1$ [Expanding along C_3]

$$\therefore\ \text{adj } A = \begin{bmatrix}\cos x & -\sin x & 0\\ \sin x & \cos x & 0\\ 0 & 0 & 1\end{bmatrix}^T\begin{bmatrix}\cos x & \sin x & 0\\ -\sin x & \cos x & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\therefore\ [f(x)]^{-1} = A^{-1} = \frac{\text{adj } A}{|A|} = \begin{bmatrix}\cos x & \sin x & 0\\ -\sin x & \cos x & 0\\ 0 & 0 & 1\end{bmatrix} \quad \text{...(ii)}$$

From (i) and (ii), we get $[f(x)]^{-1} = f(-x)$.

11. Let the investment in the first bond = ₹ x.
Then, the investment in the second bond
= ₹ (30,000 – x).
Rate of interest on first bond = 5%
Rate of interest on second bond = 7%
Let the investments represent the row matrix;
$[x \quad 30{,}000 - x]$

And rate of interest represent the column matrix

$$\begin{bmatrix} \frac{5}{100} \\ \frac{7}{100} \end{bmatrix}$$

The total interest may be represented by the product

$$[x \quad 30,000 - x]\begin{bmatrix} \frac{5}{100} \\ \frac{7}{100} \end{bmatrix}$$

(i) Total interest = ₹ 1800

$$\therefore \quad [x \quad 30,000 - x]\begin{bmatrix} \frac{5}{100} \\ \frac{7}{100} \end{bmatrix} = 1800$$

$$\Rightarrow \quad \frac{5x}{100} + \frac{(30,000 - x)7}{100} = 1800$$

$$\Rightarrow \quad \frac{5x}{100} + 2100 - \frac{7x}{100} = 1800$$

$$\Rightarrow \quad \frac{-2x}{100} = -300 \Rightarrow \quad 2x = 30,000$$

$$\Rightarrow \quad x = 15,000$$

Hence, investment in the first bond = ₹ 15000
and investment in the second bond
= ₹ (30,000 – 15,000) = ₹ 15, 000

(ii) Total interest = ₹ 2000

$$\therefore \quad [x \quad 30,000 - x]\begin{bmatrix} \frac{5}{100} \\ \frac{7}{100} \end{bmatrix} = 2000$$

$$\Rightarrow \quad \frac{5x}{100} + \frac{(30,000 - x)7}{100} = 2000$$

$$\Rightarrow \quad \frac{5x}{100} + 2100 - \frac{7x}{100} = 2000$$

$$\Rightarrow \quad -\frac{2x}{100} = -100$$

$$\Rightarrow \quad x = 5000$$

Hence, investment in the first bond = ₹ 5000
and investment in the second bond
= ₹ (30,000 – 5,000) = ₹ 25,000

12. We have

$$A^2 = AA = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$\therefore \quad A^2 - 4A - 5I = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Since $A^2 - 4A - 5I = O$

Multiplying both sides by A^{-1}, we get

$$A^{-1}A^2 - 4A^{-1}A - 5A^{-1}I = A^{-1}O$$

$$\Rightarrow \quad (A^{-1}A)\,A - 4I - 5A^{-1} = O$$

$$\Rightarrow \quad IA - 4I = 0 + 5A^{-1}$$

$$\Rightarrow \quad 5A^{-1} = A - 4I = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

13. Let the given determinant be Δ. Then

$$\Delta = \begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix}$$

$$= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + (xyz)\cdot\begin{vmatrix} 1 & x & x^3 \\ 1 & y & y^3 \\ 1 & z & z^3 \end{vmatrix}$$

$$= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + (xyz)(-1)^2\cdot\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix}$$

[interchanging the columns of 2nd det. twice]

$$= (1+xyz)\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$= \begin{vmatrix} x & x^2 & 1 \\ (y-x) & (y^2-x^2) & 0 \\ (z-x) & (z^2-x^2) & 0 \end{vmatrix}$$

$$= (1+xyz)(y-x)(z-x)\begin{vmatrix} x & x^2 & 1 \\ y & (y+x) & 0 \\ z & (z+x) & 0 \end{vmatrix}$$

Expanding along C_3

$$= (1+xyz)(y-x)(z-x)\cdot 1\cdot \begin{vmatrix} 1 & y+x \\ 1 & z+x \end{vmatrix}$$

$= (1 + xyz)\,(y - x)\,(z - x)\,(z - y)$.

$\because \Delta = 0 \Rightarrow (1 + xyz)\,(y - x)\,(z - x)\,(z - y) = 0$

$\Rightarrow (1 + xyz) = 0\ [\because\ (y - x) \neq 0, (z - x) \neq 0, (z - y) \neq 0,]$

Hence, $xyz = -1$

14. Let $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$, then

$$|A| = \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 4 - 3 = 1 \neq 0$$

and $|B| = \begin{vmatrix} -3 & 2 \\ 5 & -3 \end{vmatrix} = 9 - 10 = -1 \neq 0$

So, A and B are non-singular and invertible matrices.

$$\therefore \quad \text{adj } A = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

and $A^{-1} = \dfrac{1}{|A|}\text{adj } A = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$ $[\because |A| = 1]$ ... (i)

$$\therefore \quad \text{adj } B = \begin{bmatrix} -3 & -2 \\ -5 & -3 \end{bmatrix}$$

and $B^{-1} = \dfrac{1}{|B|}\text{adj } B = \dfrac{1}{(-1)}\begin{bmatrix} -3 & -2 \\ -5 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$ ...(ii)

$[\because \ |B| = -1]$

The given matrix equation is

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} P \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \quad i.e., \quad APB = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

$$\Rightarrow \quad A^{-1}APB = A^{-1}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

$$\Rightarrow \quad (A^{-1}A)\,(PB) = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \quad \text{[Using (i)]}$$

$$\Rightarrow \quad I(PB) = \begin{bmatrix} 2-2 & 4+1 \\ -3+4 & -6-2 \end{bmatrix}$$

$$\Rightarrow \quad PB = \begin{bmatrix} 0 & 5 \\ 1 & -8 \end{bmatrix} \Rightarrow \quad (PB)\,B^{-1} = \begin{bmatrix} 0 & 5 \\ 1 & -8 \end{bmatrix} B^{-1}$$

$$\Rightarrow \quad P(BB^{-1}) = \begin{bmatrix} 0 & 5 \\ 1 & -8 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} \quad \text{[Using (ii)]}$$

$$\Rightarrow \quad PI = \begin{bmatrix} 0+25 & 0+15 \\ 3-40 & 2-24 \end{bmatrix} \Rightarrow \quad P = \begin{bmatrix} 25 & 15 \\ -37 & -22 \end{bmatrix}$$

15. The given system of equations can be written as

$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$, i.e., $AX = B$, where

$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$

Here, $\det A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} = 0$

$\therefore$ Nothing can be said about consistency. We proceed to find adj A.

Cofactors of elements of first row are

$$(-1)^{1+1}\begin{vmatrix} 2 & 2 \\ 3 & 3 \end{vmatrix}, (-1)^{1+2}\begin{vmatrix} 2 & 2 \\ 3 & 3 \end{vmatrix}, (-1)^{1+3}\begin{vmatrix} 2 & 2 \\ 3 & 3 \end{vmatrix}, i.e., 0, 0, 0.$$

Cofactors of elements of second row are

$$(-1)^{2+1}\begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix}, (-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix}, (-1)^{2+3}\begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix}, i.e., 0, 0, 0.$$

Cofactors of elements of third row are

$$(-1)^{3+1}\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}, (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}, (-1)^{3+3}\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}, i.e., 0, 0, 0.$$

$$\therefore \text{adj } A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow (\text{adj } A)B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0$$

$\therefore$ The given system of equations may have infinitely many solutions. From the last two equations, we observe that

$2x + 2y = 2 - 2z \Leftrightarrow x + y = 1 - z$

and $3x + 3y = 4 - 3z \Leftrightarrow x + y = \dfrac{4}{3} - z$

$\Rightarrow 1 - z = \dfrac{4}{3} - z \Rightarrow 1 = \dfrac{4}{3}$, which is absurd.

Hence, the given system of equations is inconsistent and no set of values of x, y, z can satisfy all the equations.

■■

WBJEE MOCK TEST PAPER

UNIT-II : TRIGONOMETRY

The entire syllabus of Mathematics of WB-JEE is being divided in to six units, on each unit there will be a Mock Test Paper (MTP) which will be published in the subsequent issue.

CATEGORY - I

For each correct answer one mark will be awarded, whereas, for each wrong answer, 25% of total marks (1/4) will be deducted. If candidates mark more than one answer, negative marking will be done.

1. The value of $\left(1+\cos\frac{\pi}{6}\right)\left(1+\cos\frac{\pi}{3}\right)\left(1+\cos\frac{2\pi}{3}\right)\left(1+\cos\frac{7\pi}{6}\right)$ is

(a) 3/16 (b) 3/8 (c) 3/4 (d) 1/2

2. If $0 \le x \le \pi$, $0 \le y \le \pi$ and $\sin x + \sin y = 2$, then the value of $\cos(x + y)$ is

(a) -1 (b) 0 (c) 1 (d) none of these

3. If $0 < x < y < \frac{\pi}{2}$, then

(a) $x - \cos x > y - \cos y$ (b) $x - \cos x < y - \cos y$
(c) $y - \sin x < x - \sin y$ (d) none of these

4. If θ and ϕ are acute angles satisfying $\sin\theta = \frac{1}{2}$, $\cos\phi = \frac{1}{3}$, then

(a) $\frac{\pi}{3} < \theta + \phi \le \frac{\pi}{2}$ (b) $\frac{\pi}{2} < \theta + \phi < \frac{2\pi}{3}$

(c) $\frac{2\pi}{3} < \theta + \phi < \frac{5\pi}{6}$ (d) $\frac{5\pi}{6} < \theta + \phi < \pi$

5. If $\tan\theta = 5\tan\phi$, then the maximum value of $\tan^2(\theta - \phi)$ is

(a) 4/5 (b) 4/9 (c) 16/25 (d) 1

6. If $\sin x + \sin y = \sqrt{3}(\cos y - \cos x)$, then $\sin 3x + \sin 3y$ is equal to

(a) 1 (b) 0 (c) $2\sin 3x$ (d) -1

7. If $\cos\alpha + \cos(\theta + \alpha) = \cos\frac{\pi}{7}$ and $\sin\alpha + \sin(\theta + \alpha) = \sin\frac{\pi}{7}$, then the value of θ is

(a) $\frac{\pi}{3}$ (b) $\frac{2\pi}{3}$ (c) $\frac{2\pi}{7}$ (d) $\frac{5\pi}{6}$

8. If $\sin x + \cos y = \frac{1}{3}$ and $\cos x + \sin y = \frac{1}{2}$, then $\tan\frac{x-y}{2} =$

(a) -5 (b) $-1/5$ (c) 1/5 (d) 5

9. In a triangle PQR, $\angle R = \frac{\pi}{2}$, if $\tan\left(\frac{P}{2}\right)$ and $\tan\left(\frac{Q}{2}\right)$ are the roots of the equation $ax^2 + bx + c = 0$, $a \ne 0$ then

(a) $a = b + c$ (b) $c = a + b$
(c) $b = c$ (d) $b = a + c$

10. The value of $\cos 15^\circ \cos 7\frac{1^\circ}{2} \sin 7\frac{1^\circ}{2}$ is

(a) 1/2 (b) 1/8 (c) 1/4 (d) 1/16

11. If $P = \frac{1}{2}\sin^2\theta + \frac{1}{3}\cos^2\theta$, then

(a) $\frac{1}{3} \le P \le \frac{1}{2}$ (b) $P \ge \frac{1}{2}$

(c) $2 \le P \le 3$ (d) $-\frac{\sqrt{13}}{6} \le P \le \frac{\sqrt{13}}{6}$

12. Simplest form of $\frac{2}{\sqrt{2+\sqrt{2+\sqrt{2+2\cos 4x}}}}$ is

(a) $\sec\frac{x}{2}$ (b) $\sec x$ (c) $\operatorname{cosec} x$ (d) 1

13. If $(1+\sqrt{1+\alpha})\tan\alpha = 1+\sqrt{1-\alpha}$, then $\sin 4\alpha =$

(a) 1 (b) α (c) 2α (d) 4α

14. If $p = \sin^6 x + \cos^6 x$, then

(a) $\frac{3}{4} \le p \le 1$ (b) $\frac{1}{4} \le p \le 1$

(c) $\frac{3}{8} \le p \le \frac{1}{2}$ (d) none of these

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15. $\dfrac{\sqrt{2}-\sin x-\cos x}{\sin x-\cos x}=$

(a) $\tan\left(\dfrac{x}{2}-\dfrac{\pi}{8}\right)$ (b) $\tan\left(\dfrac{x}{2}+\dfrac{\pi}{8}\right)$

(c) $\tan\left(\dfrac{x}{2}-\dfrac{\pi}{4}\right)$ (d) $\tan\left(\dfrac{x}{2}+\dfrac{\pi}{4}\right)$

16. The value of cos12° + cos84° + cos156° + cos228° is

(a) −1/2 (b) 1 (c) 1/2 (d) 1/8

17. The value of $\tan\alpha + 2\tan(2\alpha) + 4\tan(4\alpha) + ...$ $....+ 2^{n-1}\tan(2^{n-1}\alpha) + 2^n\cot(2^n\alpha)$ is

(a) $\cot(2^n\alpha)$ (b) $2^n\tan(2^n\alpha)$

(c) 0 (d) $\cot\alpha$

18. The general solution of $\tan mx = \cot nx$ is

(a) $\dfrac{(2r+1)\pi}{2(m+n)}$ (b) $\dfrac{2r\pi}{(m+n)}$

(c) $\dfrac{(2r+1)\pi}{(m+n)}$ (d) $\dfrac{2r\pi}{(m-n)}$

[where r is any integer]

19. The smallest positive value of p for which the equation $\cos(p\sin x) = \sin(p\cos x)$ has a solution in $0 \le x \le 2\pi$ is

(a) $\dfrac{\pi}{\sqrt{2}}$ (b) $\dfrac{\pi}{2}$ (c) $\dfrac{\pi}{2\sqrt{2}}$ (d) $\dfrac{\pi}{4}$

20. If $\log_{0.5}\sin x = 1 - \log_{0.5}\cos x$, then the number of values of x in $-2\pi \le x \le 2\pi$, is

(a) 4 (b) 2 (c) 3 (d) 1

21. The principal value of $\sin^{-1}\tan\left(-\dfrac{5\pi}{4}\right)$ is

(a) $\dfrac{\pi}{4}$ (b) $-\dfrac{\pi}{4}$ (c) $\dfrac{\pi}{2}$ (d) $-\dfrac{\pi}{2}$

22. The trigonometric equation $\sin^{-1}x = 2\sin^{-1}a$, has a solution for

(a) $|a| \ge \dfrac{1}{\sqrt{2}}$ (b) $\dfrac{1}{2} \le |a| \le \dfrac{1}{\sqrt{2}}$

(c) all real values of a (d) $|a| \le \dfrac{1}{\sqrt{2}}$

23. $\sin\left[\dfrac{1}{2}\cot^{-1}\left(-\dfrac{3}{4}\right)\right]$ is equal to

(a) $\dfrac{1}{\sqrt{5}}$ (b) $\dfrac{2}{\sqrt{5}}$ (c) $-\dfrac{2}{\sqrt{5}}$ (d) $-\dfrac{1}{\sqrt{5}}$

24. If $\sin^{-1}x < \cos^{-1}x$, then

(a) $-1 \le x < \dfrac{1}{\sqrt{2}}$ (b) $0 \le x < \dfrac{1}{\sqrt{2}}$

(c) $\dfrac{1}{\sqrt{2}} < x \le 1$ (d) $-1 < x < 0$

25. Number of solutions of

$\tan^{-1}\sqrt{x^2+x} + \sin^{-1}\sqrt{x^2+x+1} = \dfrac{\pi}{2}$ is

(a) 0 (b) 1

(c) 2 (d) infinite

26. $(\sin^{-1}x)^3 + (\cos^{-1}x)^3 = \dfrac{\pi^3}{32}$, if x is equal to

(a) $\dfrac{1}{2}$ (b) $\dfrac{1}{\sqrt{2}}$ (c) $\dfrac{\sqrt{3}}{2}$ (d) 1

27. If the sides of a triangle are $3x + 4y$, $4x + 3y$ and $5x + 5y$, where $x > 0$, $y > 0$, the triangle will be

(a) right-angled (b) obtuse-angled

(c) equilateral (d) none of these

28. In a triangle ABC, if $\angle B = \dfrac{\pi}{3}$, $\angle C = \dfrac{\pi}{4}$ and D divides BC internally in the ratio 1 : 3, then $\dfrac{\sin\angle BAD}{\sin\angle CAD}$ equals

(a) $\dfrac{1}{3}$ (b) $\dfrac{1}{\sqrt{3}}$ (c) $\dfrac{1}{\sqrt{6}}$ (d) $\sqrt{\dfrac{2}{3}}$

29. The perpendicular AD to the base of a triangle divides it into segments such that BD, CD and AD are in the ratio 2 : 3 : 6. The angle A of the triangle is

(a) 90° (b) 60° (c) 45° (d) 30°

30. In a ΔABC, if $a = 2b$ and $A - B = \dfrac{\pi}{3}$, then $\angle C$ is equal to

(a) π/6 (b) π/4 (c) π/3 (d) π/2

CATEGORY-II

Every correct answer will yield 2 marks. For incorrect response, 25% of full mark (1/2) would be deducted. If candidates mark more than one answer, negative marking will be done.

31. The value of $3\left[\sin^4\left(\dfrac{3\pi}{2}-\alpha\right)+\sin^4(3\pi+\alpha)\right]$ $-2\left[\sin^6\left(\dfrac{\pi}{2}+\alpha\right)+\sin^6(5\pi-\alpha)\right]$ is equal to

(a) 2 (b) 1 (c) 0 (d) 4

32. The number of integral values of k for which the equation $7\cos x + 5\sin x = 2k + 1$ has a solution, is

(a) 8 (b) 6 (c) 7 (d) 9

33. The value of $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14}$ is

(a) 1/8 (b) 1/4 (c) 1/2 (d) 1

34. The base angles of a triangle are $\left(22\frac{1}{2}\right)^\circ$ and $\left(112\frac{1}{2}\right)^\circ$. If b is the base and h is the height of the triangle, then

(a) $b = 2h$ (b) $b = 3h$

(c) $b = (1+\sqrt{3})h$ (d) $2b = 3h$

35. If $\cos^{-1}\sqrt{p} + \cos^{-1}\sqrt{1-p} + \cos^{-1}\sqrt{1-q} = \frac{3\pi}{4}$, then the value of q is

(a) $\frac{1}{\sqrt{2}}$ (b) 1 (c) $\frac{1}{2}$ (d) $\frac{1}{3}$

CATEGORY-III

In this section more than 1 answer can be correct. Candidates will have to mark all the correct answers, for which 2 marks will be awarded. If, candidates marks one correct and one incorrect answer then no marks will be awarded. But if, candidate makes only correct, without marking any incorrect, formula below will be used to allot marks.

2 × (no. of correct response/total no. of correct options)

36. If $\sqrt{2}\cos A = \cos B + \cos^3 B$ and $\sqrt{2}\sin A = \sin B - \sin^3 B$,

then the value of $\sin(A - B)$ will be

(a) 1/2 (b) 1/3 (c) −1/2 (d) −1/3

37. The value of $\left(\frac{\cos\alpha+\cos\beta}{\sin\alpha-\sin\beta}\right)^n + \left(\frac{\sin\alpha+\sin\beta}{\cos\alpha-\cos\beta}\right)^n$ (where n is a whole number) is equal to

(a) 0 (b) $2\tan^n\frac{\alpha-\beta}{2}$

(c) $2\cot^n\frac{\alpha-\beta}{2}$ (d) $2\cot^n\frac{\alpha+\beta}{2}$

38. If $A = \tan6^\circ\tan42^\circ$ and $B = \cot66^\circ\cot78^\circ$ then,

(a) $A = B$ (b) $A^2 - 2AB + B^2 = 0$

(c) $A + B = 0$ (d) $A^2 + 2AB + B^2 = 0$

39. The solution of the equation $16^{\cos^2 x} + 16^{\sin^2 x} = 10 \left(0 < x < \frac{\pi}{2}\right)$ is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{2\pi}{3}$ (d) $\frac{5\pi}{3}$

40. In ΔABC, if $\cos\frac{A}{2} = \sqrt{\frac{b+c}{2c}}$, then

(a) Area of the triangle is $\frac{1}{2}ab$

(b) Circumradius is equal to $\frac{1}{2}c$

(c) Area of triangle is $\frac{1}{2}bc$

(d) Circumradius is equal to $\frac{1}{2}a$

SOLUTIONS

1. (a) : $\left(1+\cos\frac{\pi}{6}\right)\left(1+\cos\frac{\pi}{3}\right)\left(1+\cos\frac{2\pi}{3}\right)\left(1+\cos\frac{7\pi}{6}\right)$

$= \left(1+\cos\frac{\pi}{6}\right)\left(1+\cos\frac{\pi}{3}\right)\left(1+\cos\left(\pi-\frac{\pi}{3}\right)\right)\left(1+\cos\left(\pi+\frac{\pi}{6}\right)\right)$

$= \left(1+\cos\frac{\pi}{6}\right)\left(1+\cos\frac{\pi}{3}\right)\left(1-\cos\frac{\pi}{3}\right)\left(1-\cos\frac{\pi}{6}\right)$

$= \left(1-\cos^2\frac{\pi}{6}\right)\left(1-\cos^2\frac{\pi}{3}\right)$

$= \left\{1-\left(\frac{\sqrt{3}}{2}\right)^2\right\}\left\{1-\left(\frac{1}{2}\right)^2\right\} = \frac{1}{4}\times\frac{3}{4} = \frac{3}{16}$

2. (a) : $\because \sin x \le 1$, $\sin y \le 1$ and $\sin x + \sin y = 2$

$\therefore$ $\sin x = 1$ and $\sin y = 1$ $\therefore$ $x = 90^\circ = y$

Now, $\cos(x + y) = \cos180^\circ = -1$

3. (b) : $x < y$ $\therefore$ $\cos x > \cos y$

[$\because$ $\cos\theta$ is a decreasing function]

$\Rightarrow$ $-\cos x < -\cos y$

$\Rightarrow$ $x - \cos x < y - \cos y$

4. (b) : $\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$

$\because$ $\cos\frac{\pi}{3} = \frac{1}{2}$, $\cos\frac{\pi}{2} = 0$ and $0 < \frac{1}{3} < \frac{1}{2}$

$\therefore$ $\cos\frac{\pi}{2} < \cos\phi < \cos\frac{\pi}{3}$

or $\frac{\pi}{3} < \phi < \frac{\pi}{2}$ $\left[\because \cos\phi \text{ is decreasing in } 0 \le \phi \le \frac{\pi}{2}\right]$

$\therefore$ $\theta + \frac{\pi}{3} < \theta + \phi < \theta + \frac{\pi}{2}$

or $\frac{\pi}{6} + \frac{\pi}{3} < \theta + \phi < \frac{\pi}{6} + \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta + \phi < \frac{2\pi}{3}$

5. (a) : Given that $\tan\theta = 5\tan\phi$

Now, $\tan(\theta-\phi)=\dfrac{4\tan\phi}{1+5\tan^2\phi}$

$\Rightarrow \tan(\theta-\phi)=\dfrac{4}{\cot\phi+5\tan\phi}$

$\therefore \tan^2(\theta-\phi)=\dfrac{16}{(\cot\phi-5\tan\phi)^2+20\cot\phi\tan\phi}$

The least value of the denominator on R.H.S. is 20.

$\therefore$ Greatest value of $\tan^2(\theta-\phi)=\dfrac{16}{20}=\dfrac{4}{5}$

6. (b) : $\sin x+\sin y=\sqrt{3}(\cos y-\cos x)$

$\Rightarrow \sqrt{3}\cos x+\sin x=\sqrt{3}\cos y-\sin y$

$\Rightarrow \dfrac{\sqrt{3}}{2}\cos x+\dfrac{1}{2}\sin x=\dfrac{\sqrt{3}}{2}\cos y-\dfrac{1}{2}\sin y$

$\Rightarrow \cos\left(x-\dfrac{\pi}{6}\right)=\cos\left(\dfrac{\pi}{6}+y\right)$

$\Rightarrow x-\dfrac{\pi}{6}=\dfrac{\pi}{6}+y \Rightarrow 3x-\dfrac{\pi}{2}=\dfrac{\pi}{2}+3y$

$\Rightarrow 3x = \pi + 3y$

Now, $\sin3x + \sin3y = \sin(\pi + 3y) + \sin3y = 0$

7. (b) : $\cos\alpha + \cos(\theta + \alpha) = \cos\dfrac{\pi}{7}$ ... (i)

and $\sin\alpha + \sin(\theta + \alpha) = \sin\dfrac{\pi}{7}$ ... (ii)

$\therefore$ Squaring and adding (i) and (ii), we get

$[\cos\alpha + \cos(\theta + \alpha)]^2 + [\sin\alpha + \sin(\theta + \alpha)]^2$

$= \cos^2\dfrac{\pi}{7}+\sin^2\dfrac{\pi}{7}$

$\Rightarrow \cos^2\alpha + \cos^2(\theta + \alpha) + 2\cos\alpha\cos(\theta + \alpha)$
$+ \sin^2\alpha + \sin^2(\theta + \alpha) + 2\sin\alpha\sin(\theta + \alpha) = 1$

$\Rightarrow (\cos^2\alpha + \sin^2\alpha) + \{\cos^2(\theta + \alpha) + \sin^2(\theta + \alpha)\}$
$+ 2[\cos\alpha\cos(\theta + \alpha) + \sin\alpha\sin(\theta + \alpha)] = 1$

$\Rightarrow 2 + 2\cos\theta = 1 \Rightarrow 2\cos\theta = -1 \Rightarrow \cos\theta=-\dfrac{1}{2}$

$\therefore \cos\theta=\cos\left(\dfrac{2\pi}{3}\right) \Rightarrow \theta=\dfrac{2\pi}{3}$

8. (b) : $\sin x+\cos y=\dfrac{1}{3}$ and $\cos x+\sin y=\dfrac{1}{2}$

Now, $\sin x+\cos y+\cos x+\sin y=\dfrac{1}{3}+\dfrac{1}{2}$

$\Rightarrow (\sin x+\sin y)+(\cos x+\cos y)=\dfrac{5}{6}$ ... (i)

Also, $\sin x+\cos y-\cos x-\sin y=\dfrac{1}{3}-\dfrac{1}{2}$

$\Rightarrow (\sin x-\sin y)+(\cos y-\cos x)=-\dfrac{1}{6}$ ... (ii)

From (i), we get

$$2\sin\frac{x+y}{2}\cos\frac{x-y}{2}+2\cos\frac{x+y}{2}\cos\frac{x-y}{2}=\frac{5}{6}$$

$\Rightarrow 2\cos\left(\dfrac{x-y}{2}\right)\left[\sin\dfrac{x+y}{2}+\cos\dfrac{x+y}{2}\right]=\dfrac{5}{6}$ ... (iii)

From (ii), we get

$$2\cos\frac{x+y}{2}\sin\frac{x-y}{2}+2\sin\frac{y+x}{2}\sin\frac{x-y}{2}=-\frac{1}{6}$$

$\Rightarrow 2\sin\dfrac{x-y}{2}\left[\cos\dfrac{x+y}{2}+\sin\dfrac{x+y}{2}\right]=-\dfrac{1}{6}$ ... (iv)

Now, (iv) ÷ (iii) gives $\tan\dfrac{x-y}{2}=\dfrac{-1/6}{5/6}=-\dfrac{1}{5}$

9. (b) : Given that, $\tan\left(\dfrac{P}{2}\right)$ and $\tan\left(\dfrac{Q}{2}\right)$ are the roots of $ax^2 + bx + c = 0$, $a \neq 0$.

$\therefore \tan\left(\dfrac{P}{2}\right)+\tan\left(\dfrac{Q}{2}\right)=-\dfrac{b}{a}$

and $\tan\left(\dfrac{P}{2}\right)\tan\left(\dfrac{Q}{2}\right)=\dfrac{c}{a}$

But $\angle P + \angle Q + \angle R = 180°$

$\Rightarrow \angle P + \angle Q = 180° - 90°$ $[\because \angle R = 90°]$

$\Rightarrow \angle P + \angle Q = 90°$

$\Rightarrow \dfrac{1}{2}\angle P+\dfrac{1}{2}\angle Q=45° \Rightarrow \tan\left(\dfrac{P}{2}+\dfrac{Q}{2}\right)=\tan45°$

$\Rightarrow \dfrac{\tan\left(\dfrac{P}{2}\right)+\tan\left(\dfrac{Q}{2}\right)}{1-\tan\left(\dfrac{P}{2}\right)\tan\left(\dfrac{Q}{2}\right)}=1$

$\Rightarrow \dfrac{-\dfrac{b}{a}}{1-\dfrac{c}{a}}=1 \Rightarrow \dfrac{-b}{a}=1-\dfrac{c}{a} \Rightarrow \dfrac{c}{a}-\dfrac{b}{a}=1$

$\Rightarrow c - b = a \Rightarrow c = a + b$

10. (b) : $\cos15°\cos7\dfrac{1}{2}°\sin7\dfrac{1}{2}°$

$=\dfrac{1}{2}\cos15°\left(2\sin7\dfrac{1}{2}°\cos7\dfrac{1}{2}°\right)$

$=\dfrac{1}{2}\cos15°\sin15°=\dfrac{1}{4}(2\sin15°\cos15°)$

$=\dfrac{1}{4}\sin30°=\dfrac{1}{4}\times\dfrac{1}{2}=\dfrac{1}{8}$

11. (a) : $P=\frac{1}{2}\sin^2\theta+\frac{1}{3}\cos^2\theta$

$=\frac{1}{4}(1-\cos 2\theta)+\frac{1}{6}(1+\cos 2\theta)$

$=\frac{1}{4}+\frac{1}{6}-\left(\frac{1}{4}-\frac{1}{6}\right)\cos 2\theta=\frac{5}{12}-\frac{1}{12}\cos 2\theta$

$\therefore \quad \frac{1}{3}\le P\le\frac{1}{2} \quad [\because -1\le\cos 2\theta\le 1]$

12. (a) : $\frac{2}{\sqrt{2+\sqrt{2+\sqrt{2+2\cos 4x}}}}$

$=\frac{2}{\sqrt{2+\sqrt{2+\sqrt{2(1+\cos 4x)}}}}$

$=\frac{2}{\sqrt{2+\sqrt{2+\sqrt{4\cos^2 2x}}}}=\frac{2}{\sqrt{2+\sqrt{2+2\cos 2x}}}$

$=\frac{2}{\sqrt{2+\sqrt{2\cdot 2\cos^2 x}}}=\frac{2}{\sqrt{2+2\cos x}}$

$=\frac{2}{\sqrt{2(1+\cos x)}}=\frac{2}{\sqrt{2\cdot 2\cos^2\frac{x}{2}}}=\frac{1}{\cos\frac{x}{2}}=\sec\frac{x}{2}$

13. (b) : $(1+\sqrt{1+\alpha})\tan\alpha=1+\sqrt{1-\alpha}$

$\therefore \quad \tan\alpha=\frac{1+\sqrt{1-\alpha}}{1+\sqrt{1+\alpha}}$

$\Rightarrow \quad \tan\alpha=\frac{1+\sqrt{1-\sin 2\theta}}{1+\sqrt{1+\sin 2\theta}}$ [Putting $\alpha = \sin 2\theta$]

$=\frac{1+\cos\theta-\sin\theta}{1+\cos\theta+\sin\theta}=\frac{2\cos^2\frac{\theta}{2}-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}+2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$

$=\frac{2\cos^2\frac{\theta}{2}\left(1-\tan\frac{\theta}{2}\right)}{2\cos^2\frac{\theta}{2}\left(1+\tan\frac{\theta}{2}\right)}=\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$

$\therefore \quad \alpha=\frac{\pi}{4}-\frac{\theta}{2} \quad \Rightarrow \quad 4\alpha=\pi-2\theta$

$\therefore \quad \sin 4\alpha = \sin(\pi - 2\theta) = \sin 2\theta = \alpha$

14. (b) : $p = \sin^6 x + \cos^6 x$

$= (\sin^2 x + \cos^2 x)^3 - 3\sin^2 x\cos^2 x(\sin^2 x + \cos^2 x)$

$=1-\frac{3}{4}(4\sin^2 x\cos^2 x)=1-\frac{3}{4}\sin^2 2x$

p will be maximum when $\sin 2x = 0$.

$\therefore \quad p_{max} = 1$

p will be minimum when $\sin 2x = 1$

$\therefore \quad p_{min}=1-\frac{3}{4}=\frac{1}{4}$. Thus, $\frac{1}{4}\le p\le 1$

15. (a) : $\frac{\sqrt{2}-\sin x-\cos x}{\sin x-\cos x}=\frac{1-\left(\frac{1}{\sqrt{2}}\cos x+\frac{1}{\sqrt{2}}\sin x\right)}{\frac{1}{\sqrt{2}}\sin x-\frac{1}{\sqrt{2}}\cos x}$

$=\frac{1-\cos\left(x-\frac{\pi}{4}\right)}{\sin\left(x-\frac{\pi}{4}\right)}=\frac{2\sin^2\frac{1}{2}\left(x-\frac{\pi}{4}\right)}{2\sin\frac{1}{2}\left(x-\frac{\pi}{4}\right)\cos\frac{1}{2}\left(x-\frac{\pi}{4}\right)}$

$=\frac{\sin\left(\frac{x}{2}-\frac{\pi}{8}\right)}{\cos\left(\frac{x}{2}-\frac{\pi}{8}\right)}=\tan\left(\frac{x}{2}-\frac{\pi}{8}\right)$

16. (a) : $\cos 12° + \cos 84° + \cos 156° + \cos 228°$

$= (\cos 228° + \cos 12°) + (\cos 156° + \cos 84°)$

$= 2\cos 120° \cos 108° + 2\cos 120° \cos 36°$

$= 2\cos 120°(\cos 108° + \cos 36°)$

$= 2\cos 120°(-\sin 18° + \cos 36°)$

$=2\left(-\frac{1}{2}\right)\left(\frac{\sqrt{5}+1}{4}-\frac{\sqrt{5}-1}{4}\right)=-\frac{1}{2}$

17. (d) : $\tan\alpha + 2\tan(2\alpha) + 4\tan(4\alpha) + ...$

$... + 2^{n-1}\tan(2^{n-1}\alpha) + 2^n\cot(2^n\alpha)$

$= \cot\alpha - \cot\alpha + \tan\alpha + 2\tan(2\alpha) + 4\tan(4\alpha) + ...$

$... + 2^{n-1}\tan(2^{n-1}\alpha) + 2^n\cot(2^n\alpha)$

$= \cot\alpha - (\cot\alpha - \tan\alpha) + 2\tan(2\alpha) + ...$

$... + 2^{n-1}\tan(2^{n-1}\alpha) + 2^n\cot(2^n\alpha)$

$= \cot\alpha - 2\cot 2\alpha + 2\tan 2\alpha + ... + 2^{n-1}\tan(2^{n-1}\alpha)$

$+ 2^n\cot(2^n\alpha)$ [$\because \cot\alpha - \tan\alpha = 2\cot 2\alpha$]

.....

$= \cot\alpha - 2^{n-1}\cot(2^{n-1}\alpha) + 2^{n-1}\tan(2^{n-1}\alpha)$

$+ 2^n\cot(2^n\alpha)$

$= \cot\alpha - 2^n\cot(2^n\alpha) + 2^n\cot(2^n\alpha) = \cot\alpha$

18. (a) : $\tan mx = \cot nx \Rightarrow \frac{\sin mx}{\cos mx}=\frac{\cos nx}{\sin nx}$

$\Rightarrow \cos mx \cos nx = \sin mx \sin nx \Rightarrow \cos(m + n)x = 0$

$\Rightarrow \quad (m+n)x=(2r+1)\frac{\pi}{2}$

$\therefore \quad x=\frac{(2r+1)\pi}{2(m+n)}$, where r is any integer.

19. (c) : $\cos(p\sin x) = \sin(p\cos x)$

$\Rightarrow \quad \cos(p\sin x)=\cos\left(\frac{\pi}{2}-p\cos x\right)$

$\therefore \quad p\sin x=\frac{\pi}{2}-p\cos x \quad \Rightarrow \quad p(\sin x+\cos x)=\frac{\pi}{2}$

$$\Rightarrow\ p\left(\frac{1}{\sqrt{2}}\cos x+\frac{1}{\sqrt{2}}\sin x\right)=\frac{\pi}{2\sqrt{2}}$$

$$\Rightarrow\ p\cos\left(x-\frac{\pi}{4}\right)=\frac{\pi}{2\sqrt{2}}\ \ \therefore\ \ p=\frac{\pi}{2\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)}$$

Clearly, the least value of p is $\frac{\pi}{2\sqrt{2}}$.

20. (b) : $\log_{0.5}\sin x+\log_{0.5}\cos x=1$

$\Rightarrow \log_{0.5}(\sin x\cos x)=1$

$\Rightarrow \sin x\cos x=0.5=\frac{1}{2}$

$\Rightarrow 2\sin x\cos x=1 \Rightarrow 1-\sin 2x=0$

$\Rightarrow (\cos x-\sin x)^2=0 \Rightarrow \cos x-\sin x=0$

$\Rightarrow \tan x=1$

$\therefore\ x=\frac{\pi}{4}$ and $-2\pi+\frac{\pi}{4}$ [$\because \sin x>0, \cos x>0$]

21. (d) : $\sin^{-1}\tan\left(-\frac{5\pi}{4}\right)=\sin^{-1}\left(-\tan\frac{5\pi}{4}\right)$

$$=\sin^{-1}\left\{-\tan\left(\pi+\frac{\pi}{4}\right)\right\}=\sin^{-1}\left(-\tan\frac{\pi}{4}\right)$$

$$=\sin^{-1}(-1)\ =\sin^{-1}\sin\left(-\frac{\pi}{2}\right)=-\frac{\pi}{2}$$

22. (d) : Given, $\sin^{-1}x=2\sin^{-1}a$

We know, $-\frac{\pi}{2}\le\sin^{-1}x\le\frac{\pi}{2}$

$$\Rightarrow\ -\frac{\pi}{2}\le 2\sin^{-1}a\le\frac{\pi}{2}\ \Rightarrow\ -\frac{\pi}{4}\le\sin^{-1}a\le\frac{\pi}{4}$$

$$\Rightarrow\ \sin\left(-\frac{\pi}{4}\right)\le a\le\sin\frac{\pi}{4}$$

$$\Rightarrow\ -\frac{1}{\sqrt{2}}\le a\le\frac{1}{\sqrt{2}}\ \Rightarrow\ |a|\le\frac{1}{\sqrt{2}}$$

23. (b) : $\sin\left[\frac{1}{2}\cot^{-1}\left(-\frac{3}{4}\right)\right]=\sin\left[\frac{1}{2}\left(\pi-\cot^{-1}\frac{3}{4}\right)\right]$

$$=\cos\left(\frac{1}{2}\cot^{-1}\frac{3}{4}\right)=\frac{2}{\sqrt{5}}$$

24. (a) : $\sin^{-1}x<\cos^{-1}x$ (given)

We know that $-\frac{\pi}{2}\le\sin^{-1}x\le\frac{\pi}{2}\ \ \therefore\ -1\le x\le 1$;

But $\sin^{-1}x<\cos^{-1}x\Rightarrow 2\sin^{-1}x<\cos^{-1}x+\sin^{-1}x$

$$\Rightarrow\ 2\sin^{-1}x<\frac{\pi}{2}\ \Rightarrow\ \sin^{-1}x<\frac{\pi}{4}$$

$$\Rightarrow\ \sin(\sin^{-1}x)<\sin\left(\frac{\pi}{4}\right)\Rightarrow x<\frac{1}{\sqrt{2}}\ \ \therefore -1\le x<\frac{1}{\sqrt{2}}$$

25. (c) : $\sin^{-1}\sqrt{x^2+x+1}=\frac{\pi}{2}-\tan^{-1}\sqrt{x^2+x}$

$$\Rightarrow\ \sin^{-1}\sqrt{x^2+x+1}=\cot^{-1}\sqrt{x^2+x}$$

$$\Rightarrow\ \sin^{-1}\sqrt{x^2+x+1}=\sin^{-1}\frac{1}{\sqrt{x^2+x+1}}$$

$$\Rightarrow\ \sqrt{x^2+x+1}=\frac{1}{\sqrt{x^2+x+1}}$$

$\Rightarrow x^2+x+1=1 \Rightarrow x^2+x=0$

$\Rightarrow x(x+1)=0 \Rightarrow x=0, x=-1$

26. (b) : $(\sin^{-1}x)^3+(\cos^{-1}x)^3$

$$=a^3+\left(\frac{\pi}{2}-a\right)^3\quad[\text{Let }\sin^{-1}x=a]$$

$$=a^3+\left(\frac{\pi}{2}\right)^3-3\left(\frac{\pi}{2}\right)^2a+3\left(\frac{\pi}{2}\right)a^2-a^3$$

$$=\frac{3\pi}{2}\left(a^2-\frac{\pi}{2}a\right)+\frac{\pi^3}{8}$$

$$=\frac{3\pi}{2}\left[(a)^2-2\cdot a\cdot\frac{\pi}{4}+\left(\frac{\pi}{4}\right)^2-\left(\frac{\pi}{4}\right)^2\right]+\frac{\pi^3}{8}$$

$$=\frac{3\pi}{2}\left[\left(a-\frac{\pi}{4}\right)^2-\frac{\pi^2}{16}\right]+\frac{\pi^3}{8}$$

$$=\frac{3\pi}{2}\left(a-\frac{\pi}{4}\right)^2+\frac{\pi^3}{8}-\frac{3\pi^3}{32}=\frac{3\pi}{2}\left(a-\frac{\pi}{4}\right)^2+\frac{\pi^3}{32}$$

$$\therefore\ (\sin^{-1}x)^3+(\cos^{-1}x)^3=\frac{3\pi}{2}\left(a-\frac{\pi}{4}\right)^2+\frac{\pi^3}{32}$$

$$\because\ (\sin^{-1}x)^3+(\cos^{-1}x)^3=\frac{\pi^3}{32}$$

$$\therefore\ a=\frac{\pi}{4}\ \ i.e.\ \sin^{-1}x=\frac{\pi}{4}\ \Rightarrow\ x=\frac{1}{\sqrt{2}}$$

27. (b) : Clearly, the greatest side of the triangle is $5x+5y$

Let θ be the greatest angle of the triangle

$$\therefore\ \cos\theta=\frac{(3x+4y)^2+(4x+3y)^2-(5x+5y)^2}{2(3x+4y)(4x+3y)}$$

$$=-\frac{2xy}{2(3x+4y)(4x+3y)}<0$$

$\therefore$ θ is an obtuse angle.

$\therefore$ The triangle is obtuse-angled.

28. (c) : Given that $\angle B=\frac{\pi}{3}$, $\angle C=\frac{\pi}{4}$

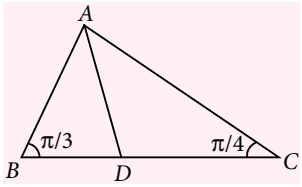
and $BD : DC = 1 : 3$

From ΔABD, we have

$$\frac{BD}{\sin\angle BAD}=\frac{AD}{\sin B}$$

From ΔCAD, we have

$$\frac{CD}{\sin\angle CAD}=\frac{AD}{\sin C}$$



$\because$ $BD : CD = 1 : 3$

$$\therefore \quad \frac{AD\sin\angle BAD}{\sin B}:\frac{AD\sin\angle CAD}{\sin C}=1:3$$

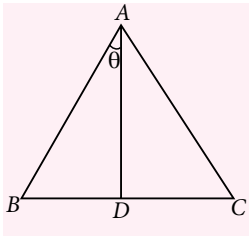
$$\Rightarrow \frac{\sin\angle BAD}{\sin\angle CAD}\cdot\frac{\sin C}{\sin B}=\frac{1}{3} \Rightarrow \frac{\sin\angle BAD}{\sin\angle CAD}=\frac{1}{3}\cdot\frac{\sin B}{\sin C}$$

$$\Rightarrow \frac{\sin\angle BAD}{\sin\angle CAD}=\frac{1}{3}\cdot\frac{\sqrt{3}/2}{1/\sqrt{2}}$$

$$\Rightarrow \frac{\sin\angle BAD}{\sin\angle CAD}=\frac{1}{3}\times\frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{1} \Rightarrow \frac{\sin\angle BAD}{\sin\angle CAD}=\frac{1}{\sqrt{6}}$$

29. (c) : Let $BD = 2x$, $CD = 3x$ and $AD = 6x$ and $\angle BAD = \theta$

Then $\angle CAD = A - \theta$



Now, $\tan\theta=\frac{BD}{AD}=\frac{2x}{6x}=\frac{1}{3}$

and $\tan(A-\theta)=\frac{CD}{AD}=\frac{3x}{6x}=\frac{1}{2}$

Also, $\tan A = \tan\{(A-\theta)+\theta\}$

$$=\frac{\tan(A-\theta)+\tan\theta}{1-\tan(A-\theta)\tan\theta}=\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{3}\cdot\frac{1}{2}}=\frac{5/6}{5/6}=1$$

$\therefore$ $\tan A = 1 \Rightarrow A = 45°$

30. (c) : In ΔABC, given that $a = 2b$ and $A - B = 60°$

Now, $\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2} \Rightarrow \tan\frac{60°}{2}=\frac{2b-b}{2b+b}\cot\frac{C}{2}$

$$\Rightarrow \frac{1}{\sqrt{3}}=\frac{1}{3}\cot\frac{C}{2} \quad \therefore \quad \cot\frac{C}{2}=\sqrt{3}=\cot 30°$$

$$\therefore \quad \frac{C}{2}=30° \Rightarrow C=60°$$

31. (b) : $3\left[\sin^4\left(\frac{3\pi}{2}-\alpha\right)+\sin^4(3\pi+\alpha)\right]$

$$-2\left[\sin^6\left(\frac{\pi}{2}+\alpha\right)+\sin^6(5\pi-\alpha)\right]$$

$= 3[\cos^4\alpha + \sin^4\alpha] - 2[\cos^6\alpha + \sin^6\alpha]$

$= 3[1 - 2\cos^2\alpha\sin^2\alpha] - 2[1 - 3\cos^2\alpha\sin^2\alpha]$

$= 3 - 2 = 1$

32. (a) : $7\cos x + 5\sin x = 2k + 1$ ... (1)

$$\Rightarrow \frac{7}{\sqrt{7^2+5^2}}\cos x+\frac{5}{\sqrt{7^2+5^2}}\sin x=\frac{2k+1}{\sqrt{7^2+5^2}}$$

Clearly (1) will have a solution if

$$\frac{2k+1}{\sqrt{7^2+5^2}}\le 1 \Rightarrow |2k+1|\le\sqrt{74}=8.6 \text{ (approx.)}$$

$\therefore$ $-8.6 \le 2k + 1 \le 8.6$

$\Rightarrow$ $-9.6 \le 2k \le 7.6 \Rightarrow -4.8 \le k \le 3.8$ ... (2)

Clearly, integral values of k satisfying the inequality (2) are $-4, -3, -2, -1, 0, 1, 2, 3$

i.e. the number of integral values of k is 8.

33. (a) : We have, $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14}$

Let $\frac{\pi}{14}=\theta$, then $14\theta=\pi \Rightarrow 7\theta=\frac{\pi}{2}$

$\therefore$ The given expression becomes

$\sin\theta\sin3\theta\sin5\theta\sin7\theta$

$$=\sin\left(\frac{\pi}{2}-6\theta\right)\sin\left(\frac{\pi}{2}-4\theta\right)\sin\left(\frac{\pi}{2}-2\theta\right)\sin\frac{\pi}{2}\left[\because 7\theta=\frac{\pi}{2}\right]$$

$= \cos6\theta\cos4\theta\cos2\theta$

$$=\frac{1}{2\sin2\theta}\cdot 2\sin2\theta\cos2\theta\cdot\cos4\theta\cos(\pi-8\theta)$$

$[\because 6\theta = 14\theta - 8\theta = \pi - 8\theta]$

$$=\frac{1}{2^2\sin2\theta}\cdot 2\sin4\theta\cos4\theta(-\cos8\theta)$$

$$=-\frac{1}{2^3\sin2\theta}\cdot 2\sin8\theta\cos8\theta=-\frac{1}{2^3\sin2\theta}\sin16\theta$$

$$=-\frac{1}{8\sin2\theta}\sin(14\theta+2\theta)=\frac{-1}{8\sin2\theta}\sin(\pi+2\theta)=\frac{1}{8}$$

34. (a) : From the triangle ABC, we get

$$\frac{a}{\sin\left(22\frac{1}{2}°\right)}=\frac{b}{\sin 45°}$$

$$\Rightarrow a=b\sqrt{2}\sin22\frac{1}{2}°$$

Again, from ΔBCD, we get

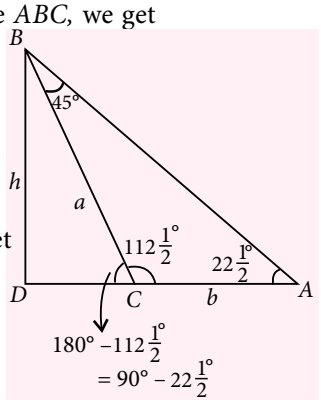
$$\frac{h}{a}=\sin\left(90°-22\frac{1}{2}°\right)$$

$$\Rightarrow h=a\cos22\frac{1}{2}°$$

$$\Rightarrow h=b\sqrt{2}\sin22\frac{1}{2}°\cos22\frac{1}{2}°$$

$$\Rightarrow h=\frac{b}{2}\cdot\sqrt{2}\sin\left(2\times22\frac{1}{2}°\right)=\frac{b}{2}\cdot\sqrt{2}\sin45°$$

$$\Rightarrow h=\frac{b}{2} \quad \therefore \quad b=2h$$



35. (c) : Let $\cos^{-1}\sqrt{1-p}=\theta$ ... (i)

$\therefore \quad \cos\theta=\sqrt{1-p} \Rightarrow \cos^2\theta=1-p$

$\Rightarrow 1-\cos^2\theta=p \Rightarrow \sin^2\theta=p$

$\therefore \quad \sin\theta=\sqrt{p}$ and $\theta=\sin^{-1}\sqrt{p}$ ... (ii)

From (i) and (ii), we get

$\cos^{-1}\sqrt{1-p}=\sin^{-1}\sqrt{p}$ ... (iii)

Now, the given equation is

$$\cos^{-1}\sqrt{p}+\cos^{-1}\sqrt{1-p}+\cos^{-1}\sqrt{1-q}=\frac{3\pi}{4}$$

$$\Rightarrow \cos^{-1}\sqrt{p}+\sin^{-1}\sqrt{p}+\cos^{-1}\sqrt{1-q}=\frac{3\pi}{4} \text{ [From(iii)]}$$

$$\Rightarrow \cos^{-1}\sqrt{1-q}=\frac{3\pi}{4}-\frac{\pi}{2}=\frac{3\pi-2\pi}{4}=\frac{\pi}{4}$$

$$\therefore \cos^{-1}\sqrt{1-q}=\frac{\pi}{4} \Rightarrow \sqrt{1-q}=\frac{1}{\sqrt{2}}$$

$$\Rightarrow 1-q=\frac{1}{2} \quad \therefore \quad q=\frac{1}{2}$$

36. (b, d) : We have, $\sqrt{2}\cos A=\cos B+\cos^3 B$ ... (i)

and $\sqrt{2}\sin A=\sin B-\sin^3 B$ ... (ii)

Squaring and adding (i) and (ii), we get

$2(\cos^2A + \sin^2A) = \cos^2B + \cos^6B + 2\cos^4B + \sin^2B + \sin^6B - 2\sin^4B$

$\Rightarrow 2 = 1 + (\cos^2B + \sin^2B)^3 - 3\cos^2B\sin^2B + 2(\cos^2B + \sin^2B)(\cos^2B - \sin^2B)$

$\Rightarrow 0 = -3\cos^2B(1 - \cos^2B) + 2(2\cos^2B - 1)$

$\Rightarrow 0 = -3\cos^2B + 3\cos^4B + 4\cos^2B - 2$

$\Rightarrow 3\cos^4B + \cos^2B - 2 = 0$

$\Rightarrow 3\cos^4B + 3\cos^2B - 2\cos^2B - 2 = 0$

$\Rightarrow (3\cos^2B - 2)(\cos^2B + 1) = 0$

$\therefore \quad \cos^2B=\frac{2}{3} \quad [\because \cos^2 B \neq -1]$

Now, $\sin^2B=1-\frac{2}{3}=\frac{1}{3}$

We have, $\cos A=\frac{\cos B(1+\cos^2 B)}{\sqrt{2}}=\frac{\pm\frac{\sqrt{2}}{\sqrt{3}}\left(1+\frac{2}{3}\right)}{\sqrt{2}}$

$$\cos A=\pm\frac{5}{3\sqrt{3}} \quad \therefore \quad \sin A=\pm\frac{\sqrt{2}}{3\sqrt{3}}$$

Now, $\sin^2(A - B) = (\sin A\cos B - \cos A\sin B)^2$

$= \sin^2A\cos^2B + \cos^2A\sin^2B - 2\sin A\cos A\sin B\cos B$

$$=\frac{2}{27}\times\frac{2}{3}+\frac{25}{27}\times\frac{1}{3}-2\times\frac{\sqrt{2}}{3\sqrt{3}}\times\frac{5}{3\sqrt{3}}\times\frac{\sqrt{2}}{\sqrt{3}}\times\frac{1}{\sqrt{3}}$$

$$=\frac{4+25}{81}-\frac{2\times2\times5}{9\times3\times3}=\frac{4+25-20}{81}=\frac{9}{81}=\frac{1}{9}$$

$\therefore \quad \sin(A-B)=\pm\frac{1}{3}$

37. (a, c) : $\left(\frac{\cos\alpha+\cos\beta}{\sin\alpha-\sin\beta}\right)^n+\left(\frac{\sin\alpha+\sin\beta}{\cos\alpha-\cos\beta}\right)^n$

$$=\left(\frac{2\cos\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}}{2\cos\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2}}\right)^n+\left(\frac{2\sin\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}}{-2\sin\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2}}\right)^n$$

$$=\cot^n\frac{\alpha-\beta}{2}+(-1)^n\cot^n\frac{\alpha-\beta}{2}$$

$$=\begin{cases}0, & \text{when } n \text{ is odd}\\ 2\cot^n\frac{\alpha-\beta}{2}, & \text{when } n \text{ is even}\end{cases}$$

38. (a, b) : Given that A = tan6° tan42° and B = cot66° cot78°

Now, $\frac{A}{B}=\frac{\tan6^\circ\tan42^\circ}{\cot66^\circ\cot78^\circ}=$ tan6° tan42° tan66° tan78°

$$=\frac{(2\sin6^\circ\sin66^\circ)(2\sin42^\circ\sin78^\circ)}{(2\cos6^\circ\cos66^\circ)(2\cos42^\circ\cos78^\circ)}$$

$$=\frac{(\cos60^\circ-\cos72^\circ)(\cos36^\circ-\cos120^\circ)}{(\cos60^\circ+\cos72^\circ)(\cos36^\circ+\cos120^\circ)}$$

$$=\frac{\left(\frac{1}{2}-\sin18^\circ\right)\left(\cos36^\circ+\frac{1}{2}\right)}{\left(\frac{1}{2}+\sin18^\circ\right)\left(\cos36^\circ-\frac{1}{2}\right)}$$

$$=\frac{\left(\frac{1}{2}-\frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{5}+1}{4}+\frac{1}{2}\right)}{\left(\frac{1}{2}+\frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{5}+1}{4}-\frac{1}{2}\right)}=\frac{(3-\sqrt{5})(3+\sqrt{5})}{(\sqrt{5}+1)(\sqrt{5}-1)}$$

$$=\frac{9-5}{5-1}=\frac{4}{4}=1 \quad \therefore \quad \frac{A}{B}=1 \Rightarrow A=B$$

$\because \quad A = B \quad \therefore \quad A^2 - 2AB + B^2 = A^2 - 2A^2 + A^2 = 0$

39. (a, b, c, d) : $16^{\sin^2x}+16^{\cos^2x}=10$

$\Rightarrow 16^{\sin^2x}+16^{1-\sin^2x}=10$

$\Rightarrow a+\frac{16}{a}=10$ [Putting $16^{\sin^2x}=a$]

$\Rightarrow a^2 - 10a + 16 = 0$

$\Rightarrow (a - 8)(a - 2) = 0 \quad \therefore \quad a = 8$ and $a = 2$

$\therefore \quad \sin x=\pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2}$

40. (a, b)

Team Kshatriya from VIT

"Winning isn't getting ahead of others, it's getting ahead of yourself"

A blazing sun, some fanning dust and hundreds of excited participants defined the 4-day spectacular event BAJA SAE INDIA, the largest motorsport gathering in the world.

Baja SAE INDIA is an intercollegiate design competition run by the Society of Automotive Engineers INDIA (SAE INDIA). Teams of students from various universities designed and built off-road buggies to withstand the harshest elements of a rough terrain. Each year as many as 110 Baja cars from around the world enter the event held in India. These teams come from places including US, Brazil, South Africa and Korea.

Team Kshatriya from VIT University, Vellore, Tamil Nadu, was one of the first teams that participated in BAJA SAE INDIA series since its inception. Formed in the year 2006 by a group of undergraduate students of the esteemed university, the team has been rising all these years unperturbed and now stands on the pinnacle of its rich history. It was ranked 3rd consecutively this time at the mega motorsport event held at NATRAX Facility at Indore, BAJA SAE INDIA 2015.

The team has 25 members from the first, second and third year. As quoted by the Team Captain, Mr. Nasit Malay R, "The team is readied every year on the basis of written tests comprising both technical and aptitude questions. This is followed by Personal Interviews and Individual Task Rounds. We make sure we get the best out of them and only then we handpick the ones who stand to take the legacy ahead. The role of the seniors is of great importance. They guide the new team with the help of their own experience and knowledge. This is what defines the team."

Dr. G. Viswanathan, Chancellor congratulated the team and said that VIT teams have been winning several global competitions not only this but several others. This is due to the combination of student's interest and our encouragement, both financially and motivationally.

So in future one would need not be surprised if VIT teams win all global level competitions on a daily basis.

■■

MATHS MUSING

SOLUTION SET-151

1. **(a) :** $S = 19(1^2 + 2^2 + 3^2 + ... + 20^2) - 2\sum_{r<s} rs$

$= 20(1^2 + 2^2 + 3^2 + ... + 20^2) - (1 + 2 + 3 + ... + 20)^2$

$= \frac{20^2 \cdot 21 \cdot 41}{6} - \frac{20^2 \cdot 21^2}{4} = 13300$

2. **(b) :** $I = \int_0^1 (1-x^2)^9 x^9 dx,\ x^2 = t$

$I = \int_0^1 (1-t)^9 t^4 \frac{dt}{2} = \frac{1}{2}\frac{9!\,4!}{14!}$

$\therefore \quad \frac{1}{I} = \frac{2 \cdot 14!}{9!\,4!} = \frac{2 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10}{24} = 20020$

3. **(c) :** $t^3 + 2t = 20t + 8,\ t^3 - 18t - 8 = 0$

t_1, t_2, t_3 are roots $\Rightarrow \Sigma t_1 t_2 = -18,\ t_1\, t_2\, t_3 = 8$

Circle $x^2 + y^2 + 2gx + 2fy + c = 0,\ x = t^2,\ y = 2t$

$t^4 + (4 + 2g)t^2 + 4ft + c = 0$

$\Sigma t_1\, t_2 = 4 + 2g \Rightarrow 2g = -22,$

$t_1\, t_2\, t_3 = 8 \Rightarrow 2f = -4,$

$t_1 + t_2 + t_3 = 0,\ t_4 = 0,\ c = 0$

$\therefore$ Circle is $x^2 + y^2 - 22x - 4y = 0$, radius $= 5\sqrt{5}$

4. **(c) :** $P(x_1, y_1) \Rightarrow$ chord of contact is $xx_1 + yy_1 = a^2$

$y = \left(-\frac{x_1}{y_1}\right)x + \frac{a^2}{y_1}$ touches $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\therefore$ Locus of P is $\frac{x^2}{b^2} + \frac{y^2}{a^2} = \frac{a^2}{b^2}$

5. **(d) :** $P(A) = 0.7,\ P(\bar{B}) = 0.6$

$P(A \cap B) = P(A) - 0.5 = 0.2$

$P(B/(A \cup \bar{B})) = \frac{0.2}{0.7 + 0.6 - 0.5} = \frac{1}{4}$

6. **(a, b, c, d) :** $10 < x \le 24$

$\Rightarrow 24^2 < x^2 + 100,\ x^2 \ge 476,\ x = 22, 23, 24$

$x > 24 > 10 \Rightarrow x^2 < 24^2 + 100,\ x^2 < 676$

$\therefore \quad x \le 25 \quad \therefore \quad x = 22, 23, 24, 25$

7. **(c) :** ABC is an equilateral triangle of side $\sqrt{6}$

Distance of A from BC = altitude $= \frac{\sqrt{3}}{2}\sqrt{6} = \frac{3}{\sqrt{2}}$

8. **(b) :** The plane OBC is $\begin{vmatrix} x & y & z \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix} = 0$

$\Rightarrow 5x - y - 7z = 0$

Distance of $A = \frac{18}{\sqrt{75}} = \frac{6\sqrt{3}}{5}$

9. **(5)**

$(1 - x)^{10} = C_0 - C_1 x + C_2 x^2 - ... + C_{10} x^{10}$

$(1 - x)^{10}\, x^4 = C_0 x^4 - C_1 x^5 + C_2 x^6 - ... + C_{10}\, x^{14}$

$S = \int_0^1 (1-x)^{10} \cdot x^4\, dx = \frac{10!\,4!}{15!}$

$\therefore \quad \frac{1}{S} = \frac{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11}{24} = 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$ (5 primes)

10. **(c) :** Letters CCLLUUAS

	Words		Number of words	
P.	$2+1$	$\Rightarrow$	$\binom{3}{2}\binom{4}{1} \cdot \frac{3!}{2!} = 36$	96 words
	$1+1+1$	$\Rightarrow$	$\binom{5}{3} \cdot 3! = 60$	
Q.	$2+2$	$\Rightarrow$	$\binom{3}{2} \cdot \frac{4!}{(2!)^2} = 18$	354 words
	$2+1+1$	$\Rightarrow$	$\binom{3}{1}\binom{4}{2} \cdot \frac{4!}{2!} = 216$	
	$1+1+1+1$	$\Rightarrow$	$\binom{5}{4} \cdot 4! = 120$	
R.	$2+2+1$	$\Rightarrow$	$\binom{3}{2} \cdot \binom{3}{1} \cdot \frac{5!}{2!2!} = 270$	1110 words
	$2+1+1+1$	$\Rightarrow$	$\binom{3}{1} \cdot \binom{4}{3} \cdot \frac{5!}{2!} = 720$	
	$1+1+1+1+1$	$\Rightarrow$	$5! = 120$	
S.	$2+2+2$	$\Rightarrow$	$\frac{6!}{(2!)^3} = 90$	2790 words
	$2+2+1+1$	$\Rightarrow$	$\binom{3}{2} \cdot \binom{3}{2} \cdot \frac{6!}{4} = 1620$	
	$2+1+1+1+1$	$\Rightarrow$	$\binom{3}{1} \cdot \binom{4}{4} \cdot \frac{6!}{2} = 1080$	

Solution Sender of Maths Musing

SET-150

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JEE WORK OUTS

PAPER-1

SECTION-1

This section contains EIGHT questions. The answer to each question is a SINGLE DIGIT INTEGER ranging from 0 to 9, both inclusive. For each question, darken the bubble corresponding to the correct integer in the ORS. Marking scheme : +4 If the bubble corresponding to the answer is darkened, 0 In all other cases.

1. Let $a_1, a_2, a_3, ..., a_{11}$ be real numbers satisfying $a_1 = 15$, $27 - 2a_2 > 0$ and $a_k = 2a_{k-1} - a_{k-2}$ for $k = 3, 4, ..., 11$. If $\frac{a_1^2 + a_2^2 + ... + a_{11}^2}{11} = 90$, then the value of $\frac{a_1 + a_2 + ... + a_{11}}{11}$ is equal to

2. Let $a + ar_1 + ar_1^2 + ... + \infty$ and $a + ar_2 + ar_2^2 + ... + \infty$ be two infinite series of positive numbers with the same first terms. The sum of the first series is r_1 and the sum of second series is r_2. The value of $(r_1 + r_2)$ is

3. If a, b, c are in H.P. and if $\left(\frac{a+b}{2a-b}\right)+\left(\frac{c+b}{2c-b}\right) \geq \sqrt{\lambda\sqrt{\lambda\sqrt{\lambda....\infty}}}$, then the maximum integral value of λ is

4. Let 'm' denotes the number of ways in which 4 different balls of green colour and 4 different balls of red colour can be distributed equally among 4 persons if each person has balls of the same colour and 'n' be corresponding figure when all the four persons have balls of different colour. Find $\frac{(m+n)}{132}$.

5. The polynomial $p(x) = 1 - x + x^2 - x^3 + \ldots + x^{16} - x^{17}$ can be written as a polynomial in y where $y = x + 1$, then let coefficient of y^2 be k then $\left[\frac{k}{400}\right]$ (where [.] denote greatest integer function) is

6. Number of ways in which $3A$'s and $4B$'s can be arranged in a row which reads the same backwards and forwards, is

7. The number of solutions for the equation $\log_4(2x^2 + x + 1) - \log_2(2x - 1) = 1$, is

8. Consider the equation $x^3 - ax^2 + bx - c = 0$, where $a, b, c \in Q$; $(a \neq 1)$. It is given that x_1, x_2 and x_1x_2 are the real roots of the equation. If $b + c = 2(a + 1)$ then $x_1x_2\left(\frac{a+1}{b+c}\right)$ is equal to

SECTION-2

This section contains TEN questions. Each question has FOUR options (a), (b), (c) and (d). ONE OR MORE THAN ONE of these four option(s) is (are) correct. For each question, darken the bubble(s) corresponding to all the correct option(s) in the ORS. Marking scheme : +4 If only the bubble(s) corresponding to all the correct option(s) is (are) darkened, 0 If none of the bubbles is darkened, –2 In all other cases.

9. The system of equations
$6x + 5y + \lambda z = 0$
$3x - y + 4z = 0$
$x + 2y - 3z = 0$ has
(a) only a trivial solution for $\lambda \in R$
(b) exactly one nontrivial solution for some real λ
(c) infinite number of nontrivial solutions for one value of λ
(d) only one solution for $\lambda \neq -5$

10. If α, β are the roots of the equation $8x^2 - 3x + 27 = 0$, then the value of $\left(\frac{\alpha^2}{\beta}\right)^{1/3} + \left(\frac{\beta^2}{\alpha}\right)^{1/3}$ is
(a) 1/3 (b) 1/4 (c) 7/2 (d) 4

11. If α, β are the real roots of $ax^2 + bx + c = 0$ and α^4, β^4 are the roots of $lx^2 + mx + n = 0$, then the roots of the equation $a^2lx^2 - 4aclx + 2c^2l + a^2m = 0$ are

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(a) real (b) imaginary
(c) opposite in sign (d) equal

12. Let S_1, S_2, ….. be squares such that for each $n \geq 1$, the length of a side of S_n equals the length of a diagonal of S_{n+1}. If the length of a side of S_1 is 10 cm, then for which of the following values of n is the area of S_n less than 1 square cm?
(a) 7 (b) 8
(c) 9 (d) 10

13. Let z_1, z_2, z_3 be non-zero complex numbers satisfying the equation $z^2 = i\bar{z}$. Which of the following statement(s) is/are CORRECT?
(a) The complex number having least positive argument is $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.
(b) $\sum_{k=1}^{3} \text{Amp}(z_k) = \frac{5\pi}{2}$
(c) Incentre of the triangle formed by z_1, z_2 and z_3 is (0, 0).
(d) Area of the triangle formed by z_1, z_2 and z_3 is $\frac{\sqrt{3}}{4}$.

14. If a, b, c are real numbers and z is a complex number such that $a^2 + b^2 + c^2 = 1$ and $b + ic = (1 + a)z$, then $\frac{1+iz}{1-iz}$ equals
(a) $\frac{b-ic}{1-ia}$ (b) $\frac{a+ib}{1+c}$
(c) $\frac{1-c}{a-ib}$ (d) $\frac{1+a}{b+ic}$

15. If the equations $x^2 + bx + c = 0$ and $bx^2 + cx + 1 = 0$ have a common root then
(a) $b + c + 1 = 0$
(b) $b^2 + c^2 + 1 = bc$
(c) $(b - c)^2 + (b - 1)^2 + (c - 1)^2 = 0$
(d) $b + c + 1 = bc$

16. The independent events, the chance that only the first occurs is 'a', the chance that only the second occurs is 'b' and the chance of only third is 'c'. If the chances of three events are respectively $\frac{a}{a+x}, \frac{b}{b+x}, \frac{c}{c+x}$, where x is the root of the equation
(a) $(a - x)(b - x)(c - x) = x^2$
(b) $(a + x)(b + x)(c + x) = x^2$
(c) $(x - a)(x - b)(x - c) = x^2$
(d) None of these

17. If roots of $ax^2 + 2bx + c = 0$ $(a \neq 0)$ are non-real complex and $a + c < 2b$, then
(a) $c > 0$ (b) $c < 0$
(c) $4a + c < 4b$ (d) $4a + c > 4b$

18. $\sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=1}^{j} 1$ is equal to
(a) $\frac{n(n+1)(n+2)}{6}$ (b) $\frac{n(n-1)(n-2)}{6}$
(c) ${}^nC_3$ (d) ${}^{n+2}C_3$

SECTION-3

This section contains TWO questions. Each question contains two columns, Column I and Column II. Column I has four entries (A), (B), (C) and (D), Column II has four entries (P), (Q), (R) and (S) . Match the entries in Column I with the entries in Column II. One or more entries in Column I may match with one or more entries in Column II. The ORS contains a 4 × 4 matrix whose layout will be similar to the one shown below.

(A) P Q R S
(B) P Q R S
(C) P Q R S
(D) P Q R S

For each entry in Column I, darken the bubbles of all the matching entries. For example, if entry (A) in Column I matches with entries (Q), (R) and (S), then darken these three bubbles in the ORS. Similarly, for entries (B), (C) and (D). Marking scheme : For each entry in Column I. +2 If only the bubble(s) corresponding to all the correct match(es) is (are) darkened, 0 If none of the bubbles is darkened, –1 In all other cases.

19. A is a set containing n elements. A subset P of set A is chosen at random. The set A is reconstructed by replacing the elements of the subset P. A subset Q of A is again chosen at random the probability that

Column I		Column II	
(A)	$P \cap Q = \phi$	P.	$n(3^{n-1})/4^n$
(B)	$P \cap Q$ is a singleton	Q.	$(3/4)^n$
(C)	$P \cap Q$ contain 2 elements	R.	${}^{2n}C_n / 4^n$
(D)	$\|P\| = \|Q\|$, where $\|X\|$ = number of elements in X	S.	$\frac{3^{n-2} n(n-1)}{2 \cdot 4^n}$

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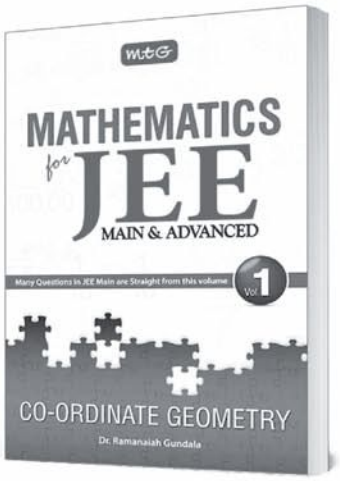

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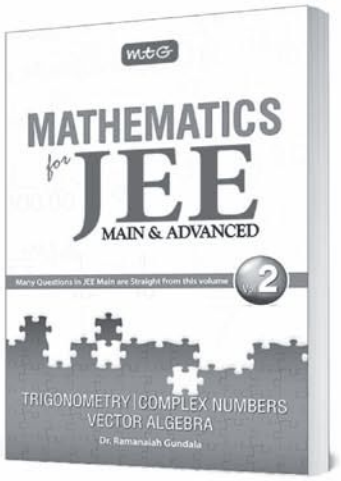

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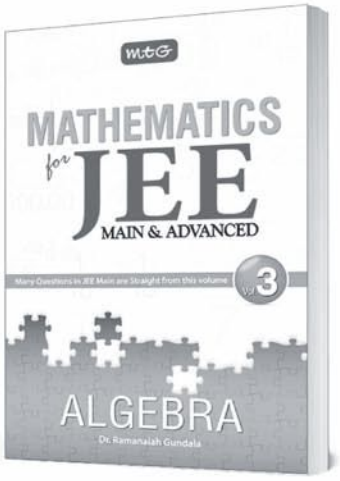

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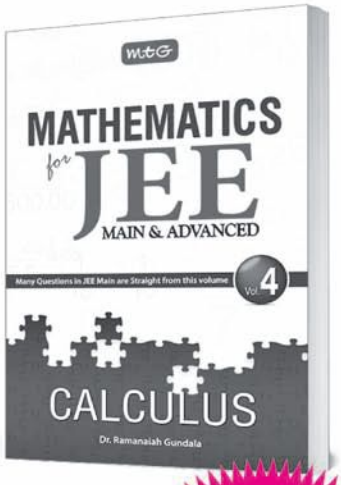

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20. Match the following:

	Column I		Column II
(A)	M_r is defined as : $M_r = \begin{bmatrix} r-1 & \frac{1}{r} \\ 1 & \frac{1}{(r-1)^2} \end{bmatrix}$ and $\lvert M_r \rvert$ is the corresponding determinant value of M_r, then $\lim\limits_{n\to\infty}(\lvert M_2 \rvert + \lvert M_3 \rvert + ... + \lvert M_n \rvert) =$	P.	0
(B	If $\begin{vmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos\gamma \\ \cos\beta & \cos\gamma & 1 \end{vmatrix} = \begin{vmatrix} 0 & \cos\alpha & \cos\beta \\ \cos\alpha & 0 & \cos\gamma \\ \cos\beta & \cos\gamma & 0 \end{vmatrix}$, then $\sin^2\alpha + \sin^2\beta + \sin^2\gamma =$	Q.	4
(C)	If $A = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$ and a matrix C is defined as $C = (BAB^{-1})(B^{-1}A^TB)$, where $\lvert B \rvert \neq 0$ and $\lvert C \rvert = k^2$, (where $k \in N$), then $k =$	R.	2
(D)	If $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ and $A^4 = -\lambda I$, then λ equals	S.	1

PAPER-2

SECTION-1

This section contains EIGHT questions. The answer to each question is a SINGLE DIGIT INTEGER ranging from 0 to 9, both inclusive. For each question, darken the bubble corresponding to the correct integer in the ORS. Marking scheme : +4 If the bubble corresponding to the answer is darkened, 0 In all other cases.

1. Let 'n' ordinary fair dice be rolled once. If the probability that at least one of the dice shows an odd number is $\frac{31}{32}$, then the value of 'n' is equal to

2. Two whole numbers are randomly selected and multiplied. If the probability that the units place in their product is "Even" is p and the probability that the units place in their product is "odd" is q, then $\left(\frac{p}{q}\right)+1$ is equal to

3. If $(y^2 - 5y + 3)(x^2 + x + 1) < 2x$ for all $x \in R$, then number of integral values of y will be

4. If in the expansion of $(1+x)^m(1-x)^n$ the coefficients of x and x^2 are 3 and –6 respectively, then 'n' is

5. What is the value of 'n' if the ratio of the 6^{th} term from the beginning to the 6^{th} term from the end of the binomial $\left(\sqrt{3}+\sqrt{2}\right)^n$ is $\sqrt{\frac{3}{2}}$?

6. The positive integer just greater than $(1 + 0.00001)^{100000}$ is

7. Sum of all the real values of x for which $\sqrt{2-x} + \frac{4}{\sqrt{2-x}+3} = 2$ is

8. If x_1, x_2, x_3 are such that,

$x_1 + x_2 + x_3 = 2$

$x_1^2 + x_2^2 + x_3^2 = 6$

$x_1^3 + x_2^3 + x_3^3 = 8$

find the value of $-(x_2 - x_3)(x_3 - x_1)(x_1 - x_2)$.

SECTION-2

This section contains EIGHT questions. Each question has FOUR options (a), (b), (c) and (d). ONE OR MORE THAN ONE of these four option(s) is(are) correct. For each question, darken the bubble(s) corresponding to all the correct option(s) in the ORS. Marking scheme : +4 If only the bubble(s) corresponding to all the correct option(s) is(are) darkened, 0 If none of the bubbles is darkened, 2 In all other cases

9. The probability that a student passes in Mathematics, Physics and Chemistry are m, p and c respectively. Of these subjects, a student has a 75% chance of passing in at least one, a 50% chance of passing in at least two and a 40% chance of passing in exactly two subjects. Which of the following relation are true?

(a) $p+m+c=\frac{19}{20}$ (b) $p+m+c=\frac{27}{20}$

(c) $pmc=\frac{1}{10}$ (d) $pmc=\frac{1}{4}$

10. If $\log_e 2$, $\log_e(2^x-1)$ and $\log_e(2^x+3)$ are in A.P., then

(a) 2^x is rational (b) x is irrational

(c) $(\sqrt{2})^x$ is irrational (d) 2^{x^2} is rational

11. If α, β are the roots of the quadratic equation $ax^2+bx+c=0$ then which of the following expression will be the symmetric function of roots?

(a) $\left|\ln\frac{\alpha}{\beta}\right|$ (b) $\alpha^2\beta^5+\beta^2\alpha^5$

(c) $\tan(\alpha-\beta)$ (d) $\left(\ln\frac{1}{\alpha}\right)^2+(\ln\beta)^2$

12. If one of the root of the equation $4x^2-15x+4p=0$ is the square of the other then the p is

(a) $\frac{125}{64}$ (b) $-\frac{27}{8}$ (c) $-\frac{125}{8}$ (d) $\frac{27}{8}$

13. If $|z_1+z_2|^2=|z_1|^2+|z_2|^2$ then

(a) $\frac{z_1}{z_2}$ is purely real

(b) $\frac{z_1}{z_2}$ is purely imaginary

(c) $z_1\bar{z}_2+z_2\bar{z}_1=0$

(d) $\text{amp}\left(\frac{z_1}{z_2}\right)$ may be equal to $\frac{\pi}{2}$

14. Let $\Delta=\begin{vmatrix} a & a^2 & 0 \\ 1 & 2a+b & (a+b)^2 \\ 0 & 1 & 2a+3b \end{vmatrix}$, then

(a) $a+b$ is a factor of Δ

(b) $a+2b$ is a factor of Δ

(c) $2a+3b$ is a factor of Δ

(d) a^2 is a factor of Δ

15. Given, $z_1+z_2+z_3=A$, $z_1+z_2\omega+z_3\omega^2=B$, $z_1+z_2\omega^2+z_3\omega=C$, where ω is a cube root of unity, then

(a) $z_1=\frac{A\omega+B\omega^2+C}{3}$

(b) $z_1z_2z_3=\frac{A^3+B^3+C^3}{3}$

(c) $z_1z_2z_3=\frac{A^3+B^3+C^3}{27}$

(d) $z_2=\frac{A+B\omega^2+C\omega}{3}$

16. If the conjugate of $(x+iy)(1-2i)$ be $1+i$, then

(a) $x=\frac{1}{5}$ (b) $x+iy=\frac{1}{5}(3+i)$

(c) $x-iy=\frac{1}{5}(3+i)$ (d) $x+iy=\frac{1-i}{1-2i}$

SECTION-3

This section contains TWO paragraphs. Based on each paragraph, there will be TWO questions. Each question has FOUR options (a), (b), (c) and (d). ONE OR MORE THAN ONE of these four option(s) is(are) correct. For each question, darken the bubble(s) corresponding to all the correct option(s) in the ORS. Marking scheme : +4 If only the bubble(s) corresponding to all the correct option(s) is(are) darkened, 0 If none of the bubbles is darkened, –2 In all other cases.

Paragraph for Q. No. 17 & 18

Suppose p is the first of n ($n>1$) AM's between two real distinct numbers a and b; q is the first of n HM's between the same two numbers.

17. The value of p is

(a) $\frac{na+b}{n+1}$ (b) $\frac{na+b}{n^2+1}$ (c) $\frac{nb+a}{n+1}$ (d) $\frac{nb+a}{n(n+1)}$

18. The value of q is

(a) $\frac{ab(n+1)}{b+an}$ (b) $\frac{ab(n+1)}{a+bn}$

(c) $\frac{ab(n^2+1)}{b+an}$ (d) $\frac{ab(n(n+1))}{a+bn}$

Paragraph for Q. No. 19 & 20

There are three pots and four coins. All these coins are to be distributed into these pots where any pot can contain any number of coins.

19. In how many ways all these coins can be distributed if all coins are identical but all pots are different?

(a) 15 (b) 16 (c) 17 (d) 81

20. In how many ways all these coins can be distributed if all coins are different but all pots are identical ?

(a) 14 (b) 21

(c) 27 (d) none of these

ANSWERS KEY

PAPER-1

1.	(0)	2.	(1)	3.	(4)	4.	(6)	5.	(2)
6.	(3)	7.	(1)	8.	(1)	9.	(c, d)	10.	(b)
11.	(a, c)	12.	(b, c, d)			13.	(a, c)	14.	(b, c)
15.	(a, c)	16.	(b)	17.	(b, c)	18.	(a, d)		

19. (A) → Q; (B) → P; (C) → S; (D) → R

20. (A) → S ; (B) → R ; (C) → Q ; (D) → Q

PAPER-2

1.	(5)	2.	(4)	3.	(2)	4.	(9)	5.	(9)
6.	(3)	7.	(1)	8.	(6)	9.	(b, c)	10.	(a, b, c)
11.	(a, b, d)			12.	(c, d)	13.	(b, c, d)		
14.	(a, b)	15.	(c, d)	16.	(b, d)	17.	(a)	18.	(b)
19.	(a)	20.	(a)						

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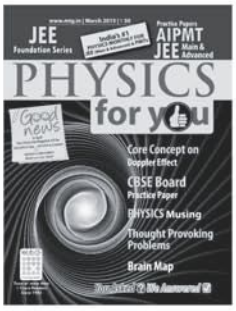

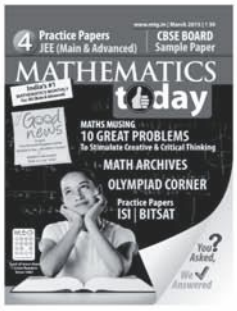

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